PROBLEM SET 4: PRECISE CONCEPTS OF LIMIT AND CONTINUITY

Note: Most of the problems were taken from the textbook [1].

Problem 1. *Prove the following statements using the $\epsilon - \delta$ definition of limit.*

a) $\lim_{x\to 1} \frac{2+4x}{3} = 2$;

b) $\lim_{x\to a} x = a$;

c) $\lim_{x\to 0} x^3 = 0$;

d) $\lim_{x\to 2}(x^2 - 4x + 5) = 1$;

e) $\lim_{x\to a} \sqrt{x} = \sqrt{a}$.

Problem 2. *If the function f is defined by*

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases},$$

then prove that $\lim_{x\to 0} f(x)$ does not exist.

Problem 3. *Suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = c$, where $c > 0$. Prove that*

$$\lim_{x\to a}[f(x) + g(x)] = \infty \quad \text{and} \quad \lim_{x\to a}[f(x)g(x)] = \infty.$$

Problem 4. *Why are the following functions continuous at every point of its domain. State the domain.*

a) $F(x) = \sin(\cos(\sin x))$;

b) $B(x) = \frac{\tan x}{\sqrt{4-x^2}}$;

c) $N(x) = \tan^{-1}(1 + e^{-x^2})$.

Problem 5. *Use continuity to evaluate $\lim_{x\to 2} x\sqrt{20 - x^2}$ and $\lim_{x\to 4} 3^{\sqrt{x^2-2x-4}}$.*

Problem 6. *Sketch the graph of the following function:*

$$f(x) = \begin{cases} 1 - x^2 & \text{if } x < 1 \\ 1/x & \text{if } x \geq 1 \end{cases}.$$

Is the function f continuous everywhere? Explain.

Problem 7. *If* $f(x) = x^2 + 10\sin x$*, show that there is a number* c *such that* $f(c) = 1000$*.*

Problem 8. *Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.*

a) $x^2 + x - 3 = 0, \quad (1, 2)$*;*

b) $\sqrt[3]{x} = 1 - x, \quad (0, 1)$*;*

c) $\sin x = x^2 - x, \quad (1, 2)$*.*

Problem 9. *Prove that each of the following equations has at least one real root.*

a) $\cos x = x^3$*;*

b) $x^5 - x^2 + 2x + 3 = 0$*;*

c) $x^5 - x^2 - 4 = 0$*.*

References

[1] J. Stewart: *Single Variable Calculus* 8th Edition, Cengage Learning, Boston 2015.