On Realizable Delta Sets of Block Monoids of Finite Cyclic Groups

Felix Gotti
felixgotti@ufl.edu

University of Florida

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Basic Notations: Free Abelian Monoid

For $n \in \mathbb{N}$, we denote the cyclic group of order n by $\mathbb{Z}_n$, and write a generic element of $\mathbb{Z}_n$ as follows:

$$[k] := \{ z \in \mathbb{Z} : n \mid z - k \}.$$

Definition of $\mathcal{F}(\mathbb{Z}_n)$

For a given $n \in \mathbb{N}$,

$$\mathcal{F}(\mathbb{Z}_n) := \left\{ \prod_{k=1}^{n-1} [k]^{\alpha_k} : \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \mathbb{N}_0 \right\}$$

will denote the free abelian monoid on $\mathbb{Z}_n \setminus \{[0]\}$. We denote the identity element of $\mathcal{F}(\mathbb{Z}_n)$ by e.

Basic Notations: Block

Consider the following elements of $\mathcal{F}(\mathbb{Z}_5)$:

- $a = [1][1][3]$ and 5 divides $1 + 1 + 3 = 5$
- $c = [1]^8[2][4]^5$ and 5 divides $8 * 1 + 2 + 5 * 4 = 30$
- $d = [4]$ and 5 does NOT divides 4
- $e = [3]^2[5][4][2]$ does NOT divides $2 * 3 + 5 + 4 + 2 = 17$.

Definition of block

For a given $n \in \mathbb{N}$, we say that $x = \prod_{k=1}^{n-1} [k]^{\alpha_k} \in \mathcal{F}(\mathbb{Z}_n)$ is a *block* if $\sum_{k=1}^{n-1} \alpha_k k$ is divisible by n.

The Block Monoid

Definition of $\mathcal{B}(\mathbb{Z}_n)$

For $n \in \mathbb{N}$ define $\text{eval}\colon \mathcal{F}(\mathbb{Z}_n) \to \mathbb{Z}_n$ by

$$\text{eval}\left(\prod_{k=1}^{n-1} [k]^{\alpha_k}\right) = \sum_{k=1}^{n-1} \alpha_k [k]$$

where the addition takes place in $\mathbb{Z}_n$. The set

$$\mathcal{B}(\mathbb{Z}_n) := \{x \in \mathcal{F}(\mathbb{Z}_n) : \text{eval}(x) = [0]\}$$

is a submonoid of $\mathcal{F}(\mathbb{Z}_n)$ called the *block monoid* of the cyclic group $\mathbb{Z}_n$.

Atoms

Definition of $\mathcal{A}(\mathbb{Z}_n)$

An element $x \in \mathcal{B}(\mathbb{Z}_n) \setminus \{e\}$ is said to be an atom if $x = ab$ where $a, b \in \mathcal{B}(\mathbb{Z}_n)$ implies that either $a = e$ or $b = e$. We denote by $\mathcal{A}(\mathbb{Z}_n)$ the set of all atoms of $\mathcal{B}(\mathbb{Z}_n)$.

- The atoms of $\mathcal{B}(\mathbb{Z}_3)$ are $[1]^3, [2]^3$, and $[1][2]$.
- Notice that $[2]^2$ is NOT an atom of $\mathcal{B}(\mathbb{Z}_3)$.
- Computing $\mathcal{A}(\mathbb{Z}_n)$ gets harder when n is larger.

Listing $\mathcal{A}(\mathbb{Z}_5)$

As an example, we show the list of atoms of $\mathcal{B}(\mathbb{Z}_5)$.

- $[1]^5$
- $[1]^3[2]$
- $[1]^2[3]$
- $[1][2]^2$
- $[1][3]^3$
- $[1][4]$
- $[2]^5$
- $[2]^3[4]$
- $[2][3]$
- $[2][4]^2$
- $[3]^5$
- $[3]^2[4]$
- $[3][4]^3$
- $[4]^5$

Some Facts About Atoms

There are some basic properties of atoms that we use frequently in this project. Some of them are the following.

1. $[a][n-a] \in \mathcal{A}(\mathbb{Z}_n)$ for any $1 \leq a < n$.
2. $[a]^n \in \mathcal{A}(\mathbb{Z}_n)$ if and only if $\gcd(a,n) = 1$.
3. If $\prod_{k=1}^{n-1}[k]^{\alpha_k} \in \mathcal{A}(\mathbb{Z}_n)$ then $\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \leq n$.
4. If $\prod_{k=1}^{n-1}[k]^{\alpha_k} \in \mathcal{A}(\mathbb{Z}_n)$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = n$ then there exists $1 \leq i < n$ such that $\alpha_i = n$ and $\alpha_j = 0$ for any $j \neq i$.

Factorizations, Lengths, and Delta Sets

Let us consider the element $x = [1]^8[2][4]^5 \in \mathcal{B}(\mathbb{Z}_5)$.

Which are the possible decompositions of x as product of atoms?

- $([2][1]^3)([1][4])^5$, having 6 atoms
- $([2][1]^3)([1]^5)([4]^5)$, having 3 atoms
- $([2][4]^2)([1]^5)([1][4])^3$, having 5 atoms

Factorizations, Lengths, and Delta Sets

For any $x \in \mathcal{B}(\mathbb{Z}_n) \setminus \{[0]\}$ we denote by $\mathsf{Z}(x)$ the set of all factorizations of x as product of atoms. The elements of $\mathsf{Z}(x)$ are also called *irreducible factorizations* of x.

Definition of Length and Set of Lengths

Let $x \in \mathcal{B}(\mathbb{Z}_n)$ and $z \in \mathsf{Z}(x)$. We call *length* of z to the number of atoms that appears in z, and we denote the length of z by $|z|$. We define the *set of lengths* of x by

$$\mathsf{L}(x) = \{\, |z| : z \in \mathsf{Z}(x) \,\}.$$

Factorizations, Lengths, and Delta Sets

Given an element $x \in \mathcal{B}(\mathbb{Z}_n)$, we would like to measure how far from one to another are the lengths of the irreducible factorizations of x.

Definition of set of deltas

Let $n \in \mathbb{N}$, $x \in \mathcal{B}(\mathbb{Z}_n)$, and $\mathsf{L}(x) = \{l_1, l_2, \dots, l_k\}$ where $l_1 < l_2 < \cdots < l_k$. If $|\mathsf{L}(x)| > 1$, the *delta set* of x is defined to be the set

$$\Delta(x) = \{\, l_{i+l} - l_i : 1 \le i < k \,\}.$$

If $|\mathsf{L}(x)| = 1$ then we define $\Delta(x)$ as the empty set. In addition, we define the delta set of $\mathcal{B}(\mathbb{Z}_n)$ to be

$$\Delta(\mathbb{Z}_n) = \bigcup_{x \in \mathcal{B}(\mathbb{Z}_n)} \Delta(x).$$

Examples: Factorizations, Set of Lengths, and Delta Set

Returning to the example of $x = [1]^8[2][4]^5 \in \mathcal{B}(\mathbb{Z}_5)$, we can find now its set of lengths and its delta set.

- $\mathsf{Z}(x) = \{([2][1]^3)([1][4])^5, ([2][1]^3)([1]^5)([4]^5), ([2][4]^2)([1]^5)([1][4])^3\}$.
- $\mathsf{L}(x) = \{3, 5, 6\}$
- $\Delta(x) = \{1, 2\}$

Realizable Delta Sets and the D-set

Now we introduce the arithmetic invariant of the block monoid $\mathcal{B}(\mathbb{Z}_n)$ that we wish to study.

Definition of D-set

We say that $S \subseteq \Delta(\mathbb{Z}_n)$ is a *realizable delta set* of $\mathcal{B}(\mathbb{Z}_n)$ if there exists $x \in \mathcal{B}(\mathbb{Z}_n)$ such that $S = \Delta(x)$. Also we use the following notation for the set of all realizable delta sets of $\mathcal{B}(\mathbb{Z}_n)$:

$$\mathcal{D}(\mathbb{Z}_n) := \{ \Delta(x) : x \in \mathcal{B}(\mathbb{Z}_n) \}.$$

We say that $\mathcal{D}(\mathbb{Z}_n)$ is the *D-set* of $\mathcal{B}(\mathbb{Z}_n)$.

Motivation

The following result fully describes $\Delta(\mathbb{Z}_n)$.

Motivation Theorem

For $n \in \mathbb{N}$ we have $\Delta(\mathbb{Z}_n) = \{1, 2, \ldots, n-2\}$.

Because

- the previous theorem shows that $\Delta(\mathbb{Z}_n)$ is a very convenient subset of $\mathbb{N}$ (an interval starting at 1) and
- nothing has been study so far about $\mathcal{D}(\mathbb{Z}_n)$

it seems reasonable attempt to describe the D-set of $\mathcal{B}(\mathbb{Z}_n)$.

Project and its Importance

Project

For $n \in \mathbb{N}$, we carefully studied $\mathcal{D}(\mathbb{Z}_n)$, the D-set of $\mathcal{B}(\mathbb{Z}_n)$.

Why is important to study $\mathcal{D}(\mathbb{Z}_n)$?

- The block monoid naturally arises in Algebraic Number Theory; for example, when proving the Number Class Theorem.
- By giving a partial description of $\mathcal{D}(\mathbb{Z}_n)$, we can offer a useful tool for the study of factorizations in the commutative cancellative monoid $\mathcal{B}(\mathbb{Z}_n)$ and the Krull monoid it determines.

Realizable Delta Sets of Cardinality One

One of the inclusions of the Motivation Theorem is proved by showing that any singleton subset of $\{1, 2, \ldots, n-2\}$ is a realizable delta set of $\mathcal{B}(\mathbb{Z}_n)$.

Singleton Realizable Delta Sets

If $n \in \mathbb{N}$ then $\{j\} \in \mathcal{D}(\mathbb{Z}_n)$ for $1 \le j \le n-2$.

The above result is a piece of information we can use when describing $\mathcal{D}(\mathbb{Z}_n)$.

The Case $n = 5$

What we know about $\mathcal{D}(\mathbb{Z}_5)$?

- $\mathcal{D}(\mathbb{Z}_5) \subseteq \mathcal{P}(\Delta(\mathbb{Z}_5)) = \mathcal{P}(\{1,2,3\})$ by the Motivation Theorem.
- $\{1\}, \{2\}, \{3\} \in \mathcal{D}(\mathbb{Z}_5)$ by the previous result.
- $\{1,2\} \in \mathcal{D}(\mathbb{Z}_5)$ because $\Delta([1]^8[2][4]^5) = \{1,2\}$.

What can we say about $\{1,3\}$ and $\{1,2,3\}$?

Where does $n-2$ live?

Now we present the generalized version of the result we used to complete the description of $\mathcal{D}(\mathbb{Z}_5)$.

Main Theorem

Let $n \in \mathbb{N}$ and $x \in \mathcal{B}(\mathbb{Z}_n)$ such that $n-2 \in \Delta(x)$. Then $|\Delta(x)| = 1$ (i.e., $\Delta(x) = \{n-2\}$).

The following corollary follows immediately from the theorem.

Corollary

If $n \in \mathbb{N}$ then $\Delta(\mathbb{Z}_n) \notin \mathcal{D}(\mathbb{Z}_n)$.

Returning to the Case $n = 5$

As an application of the Main Theorem, we obtain

- $\{1, 3\} \notin \mathcal{D}(\mathbb{Z}_n)$
- $\{1, 2, 3\} \notin \mathcal{D}(\mathbb{Z}_n)$.

Therefore, we can complete the description of $\mathcal{D}(\mathbb{Z}_5)$.

Proposition

$\mathcal{D}(\mathbb{Z}_5) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$.

After seeing the description of $\mathcal{D}(\mathbb{Z}_5)$, we might think that the only realizable delta sets of $\mathcal{B}(\mathbb{Z}_n)$ have cardinality less or equal than 2, or maybe small (bounded above).

Archimedean-type Property for $\mathcal{D}(\mathbb{Z}_n)$

- By the Main Theorem, we know that $\Delta(\mathbb{Z}_n)$ is not a realizable delta set of $\mathcal{B}(\mathbb{Z}_n)$; therefore we might believe in the existence of a uniform bound M for the lengths of all elements in $\mathcal{D}(\mathbb{Z}_n)$ for all n.
- However, we could prove that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements in $\mathcal{B}(\mathbb{Z}_n)$ such that the sequence $\{|\Delta(x_n)|\}_{n\in\mathbb{N}}$ tends to infinity, as the following theorem indicates.

Theorem (Archimedean Property)

For any $M \in \mathbb{N}$ there exist $n \in \mathbb{N}$ and $x \in \mathcal{B}(\mathbb{Z}_n)$ such that $|\Delta(x)| > m$.

Future Work and Conclusions

- The previous results suggest that a full classification of $\mathcal{D}(\mathbb{Z}_n)$ might be a very arduous task. However, there are several steps that can be helpful when trying to give a deeper description of $\mathcal{D}(\mathbb{Z}_n)$.
- For example, we can try to find bounds for the cardinality of the elements of $\mathcal{D}(\mathbb{Z}_n)$.

Definition of Principal Delta

Let G be a finite abelian group. We call

$$\eta(G) = \max\{\, |S| : S \in \mathcal{D}(G)\,\}$$

the *principal delta* of $\mathcal{B}(G)$.

Future Work and Conclusions

Some observations:

- Based on the results of this project, we have

$$4 \leq \eta(\mathbb{Z}_n) \leq n - 3.$$

- However, the Archimedean Property shows that 4 is not a good lower bound. An accurate lower bound should depend on n.

Future Work

As an extension of this project, we can try to find better bounds for $\eta(\mathbb{Z}_n)$.

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