Introduction to Positroids, and Unit Interval Positroids

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Outline

1. **Positroids**
2. **Representations of Positroids**
3. **Unit Interval Positroids**

Definition of Matroids

Definition (Matroid)

Let E be a finite set, and let $\mathcal{B}$ be a nonempty collection of subsets of E. The pair $M = (E, \mathcal{B})$ is a *matroid* if for all $B, B' \in \mathcal{B}$ and $b \in B \setminus B'$, there exists $b' \in B' \setminus B$ such that $(B \setminus \{b\}) \cup \{b'\} \in \mathcal{B}$.

If $M = (E, \mathcal{B})$ is a matroid, then:

- the elements of $\mathcal{B}$ are said to be *bases* of M;
- a subset of a basic element is called *independent*.

Proposition: Bases of the same matroid have the same cardinality.

- Given a matroid M, the *rank* of M, denoted by $r(M)$, is the size of any basis.

Operations on Matroids

Let $M = (E, \mathcal{B})$ and $M' = (E', \mathcal{B}')$ be two matroids and $S \subseteq E$.

1. $M \oplus M' := (E \sqcup E', \{B \sqcup B' : B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\})$ is a matroid, which is called the *direct sum* of M and M'.
2. $M^* := (E, \{E \setminus B : B \in \mathcal{B}\})$ is a matroid, which is called the *dual* matroid of M.
3. $M|S := (S, \{B \cap S : B \in \mathcal{B} \text{ and } |B \cap S| \text{ is maximal}\})$ is a matroid, which is called *restriction* of M to S.
4. $M/S := (E \setminus S, \{B \setminus S : B \in \mathcal{B} \text{ and } |B \cap S| \text{ is maximal}\})$ is a matroid, which is called *contraction* of M by S.

Definition of Positroids

Let $\mathbb{F}$ be a field and $d, n \in \mathbb{N}$ such that $d \leq n$.

Definition

A matroid $M = ([n], \mathcal{B})$ of rank d is *representable* if there is $A \in M_{d \times n}(\mathbb{F})$ with columns $A_1, \dots, A_n$ such that $B \subseteq [n]$ is a basis of M iff $\{A_i \mid i \in B\}$ is a basis for $\mathbb{F}^d$.

- With notation as above, we say that M is *represented* by A.
- We say that a real matrix is *totally nonnegative* (TNN) if all its maximal minors are nonnegative.

Definition (Positroid)

A *positroid* on $[n]$ or rank d is a matroid that can be represented by a $d \times n$ full-rank TNN real matrix.

New Positroids from Old Ones

If $n \in \mathbb{N}$ and $\ell, m \in [n]$ the *cyclic interval* $[\ell, m]$ is

$$[\ell, m] = \begin{cases} \{\ell, \ell+1, \dots, m\} & \text{if } \ell \leq m \\ \{\ell, \ell+1, \dots, n, 1, \dots, m\} & \text{if } \ell > m \end{cases}.$$

Theorem (Ardila-Rincón-Williams)

Let $[\ell+1, m]$ and $[m+1, \ell]$ be a decomposition of $[n]$ into two cyclic intervals, and let M and M' be two positroids on the ordered ground sets $[\ell+1, m]$ and $[m+1, \ell]$, respectively. Then $M \oplus M'$ is a positroid on $[n]$.

Theorem (Ardila-Rincón-Williams)

Let M be a positroid on $[n]$, and let $S \subseteq [n]$. Then the dual, the restriction, and the contraction of M are also positroids.

Grassman Necklaces

Let $d, n \in \mathbb{N}$ such that $d \leq n$.

Definition (Grassmann Necklace)

An n-tuple $(I_1, \ldots, I_n)$ of d-subsets of $[n]$ is called a *Grassmann necklace* of *type* (d, n) if for all $i \in [n]$ the next conditions hold:

- $i \in I_i$ implies $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$ for some $j \in [n]$;
- $i \notin I_i$ implies $I_{i+1} = I_i$.

Remark: In the above definition, we assume that $I_{n+1} = I_1$.

Definition: The i-th order on $[n]$ is defined by

$$i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-2 <_i i-1.$$

For a rank d matroid M we can define $\mathcal{I}(M) = (I_1, \ldots, I_n)$, where I_i is the lexicographically $<_i$-minimal basis of M.

Grassmann Necklaces and Positroids

Proposition

For any matroid $M = ([n], \mathcal{B})$ of rank d, the sequence $\mathcal{I}(M)$ is a Grassmann necklace of type (d, n).

Theorem (Postnikov)

For $d, n \in \mathbb{N}$ such that $d \leq n$, let $\mathcal{I} = (I_1, \dots, I_n)$ be a Grassmann necklace of type (d, n). Then

$$\mathcal{B}(\mathcal{I}) = \left\{ B \in \binom{[n]}{d} \;\middle|\; I_j \prec_j B \text{ for every } j \in [n] \right\}$$

is the collection of bases of a positroid $\mathcal{P}(\mathcal{I}) = ([n], \mathcal{B}(\mathcal{I}))$, where $\prec_i$ is the Gale i-order on $\binom{[n]}{d}$. Moreover, $\mathcal{P}(\mathcal{I}(P)) = P$ for all positroids P.

A Bijective Representation Via Grassmann Necklaces

Denoting the set of all positroids on $[n]$ of rank d by $\mathcal{P}_{d,n}$, we have the following result.

Corollary

The map $\mathcal{I}\colon \mathcal{P}_{d,n} \to \mathcal{I}(\mathcal{P}_{d,n})$ is a bijection.

Decorated Permutations

Decorated permutations, also in one-to-one correspondence with positroids, will provide a more succinct representation.

Definition (Decorated Permutation)

A *decorated permutation* of $[n]$ is an element $\pi \in S_n$ whose fixed points j are marked either "clockwise"(denoted by $\pi(j) = \underline{j}$) or "counterclockwise" (denoted by $\pi(j) = \bar{j}$).

Observation: A *weak i-excedance* of a decorated permutation $\pi \in S_n$ is an index $j \in [n]$ satisfying $j <_i \pi(j)$ or $\pi(j) = \bar{j}$. It is easy to see that the number of weak i-excedances does not depend on i, so we just call it the number of *weak excedances*.

Decorated Permutations and Grassmann Necklaces

To every Grassmann necklace $\mathcal{I} = (I_1, \dots, I_n)$ one can associate a decorated permutation $\pi_{\mathcal{I}}$ as follows:

- if $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, then $\pi_{\mathcal{I}}(j) = i$;
- if $I_{i+1} = I_i$ and $i \notin I_i$, then $\pi_{\mathcal{I}}(i) = \underline{i}$;
- if $I_{i+1} = I_i$ and $i \in I_i$, then $\pi_{\mathcal{I}}(i) = \overline{i}$.

The assignment $\mathcal{I} \mapsto \pi_{\mathcal{I}}$ defines a one-to-one correspondence between the set of Grassmann necklaces of type (d, n) and the set of decorated permutations of $[n]$ having exactly d weak excedances.

Proposition

The map $\mathcal{I} \mapsto \pi_{\mathcal{I}}$ is a bijection between the set of Grassmann necklaces of type (d, n) and the set of decorated permutations of $[n]$ having exactly d weak excedances.

Postnikov's Map

Lemma (Postnikov)

For an $n \times n$ real matrix $A = (a_{i,j})$, consider the $n \times 2n$ matrix $B = \phi(A)$, where

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n} \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \overset{\phi}{\mapsto} \begin{pmatrix} 1 & \dots & 0 & 0 & \pm a_{n,1} & \dots & \pm a_{n,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & -a_{2,1} & \dots & -a_{2,n} \\ 0 & \dots & 0 & 1 & a_{1,1} & \dots & a_{1,n} \end{pmatrix}.$$

Under this correspondence, $\Delta_{I,J}(A) = \Delta_{([n]\setminus I)\cup(n+J)}(B)$ for all $I, J \subseteq [n]$ satisfying $|I| = |J|$ (here $\Delta_{I,J}(A)$ is the minor of A determined by the rows I and columns J, and $\Delta_K(B)$ is the maximal minor of B determined by columns K).

Dyck Matrices

Definition (Dyck Matrix)

A binary square matrix is said to be a *Dyck matrix* if its zero entries form a right-justified Young diagram strictly above the main diagonal and anchored in the upper-right corner.

Notation: $\mathcal{D}_n$ will denote the set of all $n \times n$ Dyck matrices.
Example: The following 6×6 matrix is a Dyck matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Observations:

- All minors of a Dyck matrix are TNN.
- $|\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.

Unit Interval Positroids

Each Dyck matrix $D \in \mathcal{D}_n$ *induces* a positroid via the Postnikov's map, namely, the positroid represented by the full-rank TNN matrix $\phi(D)$.

Definition (Unit Interval Positroid)

A positroid on $[2n]$ induced by a unit interval order is called *unit interval positroid*.

Notation: Let $\mathcal{P}_n$ denote the set of all unit interval positroids on the ground set $[2n]$, and let $\varphi\colon \mathcal{D}_n \to \mathcal{P}_n$ be the map described above.

Observations:

- The map φ is onto by definition.
- There are at most $\frac{1}{n+1}\binom{2n}{n}$ unit interval positroids on $[2n]$.

Decorated Permutation of Unit Interval Positroids

Decorated permutations of positroids in $\mathcal{P}_n$ are $2n$-cycles satisfying certain special properties.

Theorem (Chavez-G)

Decorated permutations associated to unit interval positroids on $[2n]$ are $2n$-cycles $(1\ j_1\ \dots\ j_{2n-1})$ satisfying the following two conditions:

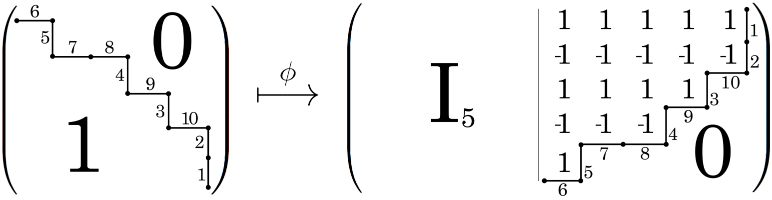
1. *in the sequence $(1, j_1, \dots, j_{2n-1})$ the elements $1, \dots, n$ appear in increasing order while the elements $n+1, \dots, 2n$ appear in decreasing order;*
2. *for every $1 \leq k \leq 2n-1$, the set $\{1, j_1, \dots, j_k\}$ contains at least as many elements of the set $\{1, \dots, n\}$ as elements of the set $\{n+1, \dots, 2n\}$.*

Decorated Permutation from Dyck Matrix

Theorem (Chavez-G)

If we number the n vertical steps of the semiorder path of A from bottom to top in increasing order with $\{1, \ldots, n\}$ *and the n horizontal steps from left to right in increasing order with* $\{n+1, \ldots, 2n\}$*, then we get the decorated permutation of the unit interval positroid induced by P by reading the semiorder path in northwest direction.*

Example: From the following Dyck matrix D (or $\phi(D)$)



we recover the decorated permutation $\pi = (1\ 2\ 10\ 3\ 9\ 4\ 8\ 7\ 5\ 6)$.

Counting Unit Interval Positroids

Theorem (Chavez-G)

The map $\varphi\colon \mathcal{D}_n \to \mathcal{P}_n$ is a bijection.

Corollary

There number of unit interval positroids on $[2n]$ is precisely the n-th Catalan number.

References

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