Incidence Algebra and Möbius Inversion Formula

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Outline

Incidence Algebras

Möbius Inversion Formula

Möbius Functions of Lattices (Examples)

Möbious Function of Distributive Lattices

Definition of Incidence Algebra

Definition

Let P be a locally finite poset and k be a field.

- $\text{Int}(P)$ denote the set of closed intervals of P.
- The *incidence algebra* of P over k, denoted by $I(P, k)$ is the k-algebra of all functions $\text{Int}(P) \to k$, where for $f, g \in \text{Int}(P)$ multiplication is defined via the convolution

$$fg(s,t) = \sum_{s \le x \le t} f(s,x)g(x,t).$$

Remarks:

1. Multiplication in $I(P, k)$ is well defined because P is locally finite.
2. $I(P, k)$ is an associative algebra with identity denoted by δ, which satisfies that $\delta(s,t) = 1$ if $s = t$, and $\delta(s,t) = 0$ otherwise.

Incidence Algebra (continuation)

Theorem

An element $f \in I(P, k)$ has a two-sided inverse iff $f(s, s) \neq 0$ for all $s \in P$.

Sketch of Proof: If $fg = \delta$ then, for $s \in P$, $f(s, s)g(s, s) = \delta(s, s) = 1$; so $f(s, s) \neq 0$ for all $s \in P$. Conversely, if $f(s, s) \neq 0$ for all $s \in P$, we define $g(s, s) = f(s, s)^{-1}$ and, inductively,

$$g(s, t) = -f(s, s)^{-1} \sum_{s<x\leq t} f(s, x)g(x, t) \quad \text{when } s < t.$$

It follows immediately that g is a right inverse of f. Similarly, we can find a left inverse h of f. By associativity of $I(P, k)$, $g = h$.

□

The Zeta Function

Definition

The *zeta function* $\zeta \in I(P, k)$ is defined by $\zeta(s, t) = 1$ for all $s, t \in P$ such that $s \leq t$.

Theorem

If $n \in \mathbb{N}$ the following holds:

1. $\zeta^n(s, t) = \sum_{s=s_0 \leq s_1 \leq \cdots \leq s_n = t} 1$, *the number of multichains of length n from s to t.*
2. $(\zeta - 1)^n(s, t) = \sum_{s=s_0 < s_1 < \cdots < s_n = t} 1$, *the number of chains of length n from s to t.*
3. $2 - \zeta$ *is invertible, and* $(2 - \zeta)^{-1}(s, t)$ *counts the number of chains from s to t.*

Sketch of Proof: (1) and (2) follows by induction. For (3), choose $n = \ell([s, t])$ in the identity

$$(2 - \zeta)^{-1}(s, t) = [1 + (\zeta - 1) + \cdots + (\zeta - 1)^n](s, t). \quad \square$$

The Möbius Function

Since $\zeta(s,s) = 1$ for all $s \in P$, it is invertible.

Definition

In a locally finite poset P, the inverse of ζ, denoted by μ, is called Möbius function.

Theorem

The Möbius function is uniquely determined by the following recurrence: $\mu(s,s) = 1$ *and* $\mu(s,t) = -\sum_{s \le x < t} \mu(s,x)$.

Proof: Exercise.

The Möbius Inversion Formula

Theorem

Let k be a field and let P be a poset whose principal order ideals are finite. For $f, g \in I(P,k)$, we have that $g(s) = \sum_{t \le s} f(t)$ for each $s \in P$ iff

$$f(s) = \sum_{t \le s} g(t)\mu(t,s) \quad \textit{for each} \quad s \in P.$$

Sketch of Proof: The algebra $I(P,k)$ acts on the right of the vector space V of functions $P \to k$ via

$$(g\varphi)(s) = \sum_{t \le s} g(t)\varphi(t,s), \text{ for } g \in V \text{ and } \varphi \in I(P,k).$$

Then theorem then can be translated to $g = f\zeta$ iff $g\mu = f$. □

The Intersection Example

Example: Let P be the poset of all possible intersection of the finite sets $S_1, \dots, S_n$. For $T \in P$ let $f(T) := |T \setminus (\cup_{T' < T} T')|$ and let $g(T) = |T|$. Note that $f(\hat{1}) = f(S_1 \cup \dots \cup S_n) = 0$. Since $g(T) = \sum_{T' \leq T} f(T')$, by the Möbius Inversion Formula:

$$f(T) = \sum_{T' \leq T} g(T')\mu(T', T).$$

Evaluating the above equation at $T = \hat{1}$, we obtain

$$0 = f(\hat{1}) = \sum_{I \in P} g(I)\mu(I, \hat{1}),$$

which translate to

$$|S_1 \cup \dots \cup S_n| = -\sum_{I} \mu(I, S_1 \cup \dots \cup S_n)|I|,$$

where I runs over all nonempty intersection of the S_i's.

The Product Theorem

Theorem (The Product Theorem)

Let P and Q be locally finite posets, and let $P \times Q$ be their direct product. If $(s, t) \leq (s', t')$ in $P \times Q$ then

$$\mu_{P\times Q}((s, t), (s', t')) = \mu_P(s, s')\mu_Q(t, t').$$

Sketch of Proof: It is enough to check that $F\colon \mathrm{Int}(P \times Q) \to k$ defined by $F((s, t), (s', t')) = \mu_P(s, s')\mu_Q(t, t')$ satisfies the defining recurrence of the Möbius function.

□

The Sieve Method from the MIF

Example 2: The Möbius function of $[n]$ is given by $\mu(i,i) = 1$, $\mu(i, i+1) = -1$, and $\mu(i,j) = 0$ if $j > i+1$.

Example 3: For $n \in \mathbb{N}$, the poset B_n is isomorphic to the boolean algebra $\mathbf{2^n}$ via $S \mapsto (s_1, \dots, s_n)$, where $s_i = 1$ iff $i \in S$. Then if $S, T \in B_n$ such that $S \subseteq T$ and $T \mapsto (t_1, \dots, t_n)$, we have

$$\begin{aligned}\mu(S,T) &= \mu((s_1,\dots,s_n),(t_1,\dots,t_n)) \\ &= \prod_{i\in[n]} \mu_{[2]}(s_i,t_i) = (-1)^{|T\setminus S|}.\end{aligned}$$

Therefore the Möbius Inversion Formula on B_n translates to the Sieve Method: $f(T) = \sum_{S\subseteq T} g(S)$ iff

$$g(T) = \sum_{S\subseteq T} \mu(S,T) g(S) = \sum_{S\subseteq T} (-1)^{|T\setminus S|} g(S).$$

The MIF in Number Theory

Example: Let $n = p_1^{n_1} \dots p_k^{n_k}$, where $n_i \in \mathbb{N}$ and $p_1, \dots, p_k$ are different primes. Then D_n is isomorphic to $\mathbf{n_1+1} \times \cdots \times \mathbf{n_k+1}$ via $n_i = p_1^{i_1} \dots p_k^{i_k} \mapsto (i_1+1, \dots, i_k+1)$. Therefore if $n_j = (j_1+1, \dots, j_k+1)$ such that n_i divides n_j, we have that

$$\begin{aligned} \mu_{D_n}(n_i, n_j) &= \mu_{D_n}((i_1+1, \dots, i_k+1), (j_1+1, \dots, j_k+1)) \\ &= \prod_{t=1}^{k} \mu_{\mathbf{n_t+1}}(i_t+1, j_t+1) \end{aligned}$$

Therefore $\mu_{D_n}(n_i, n_j) = 0$ if n_j/n_i is not squarefree and $\mu_{D_n}(n_i, n_j) = (-1)^m$, where m is the number of primes dividing n_j/n_i, if n_j/n_i is squarefree. Note that $\mu_{D_n}(n_i, n_j) = \mu(n_j/n_i)$, where μ is the standard Möbius function defined in number theory. Also, the Möbius Inversion Formula translates to the corresponding formula in number theory.

Möbius Algebra

Definition

Let L be a lattice and k be a field. The Möbius algebra, denoted by $A(L, k)$, is the k-space with basis L, with multiplication given by $s \cdot t = s \wedge t$ for all $s, t \in L$.

Theorem

Let L be a finite lattice and let $A'(L, k)$ be the algebra $\bigoplus_{s \in L} k_s$, where $k_s \cong k$ for all $s \in L$. If δ'_s is the identity of k_s then the map $\theta \colon A(L, k) \to A(L, k')$ defined by $\theta(\delta_s) = \delta'_s$ and extended by linearity is an algebra isomorphism.

Sketch of Proof: The map θ sends a basis to a basis, so it is a k-space isomorphism. Also for $s, t \in L$,

$$\theta(s)\theta(t) = \Big(\sum_{x \leq s} \delta'_x\Big)\Big(\sum_{y \leq t} \delta'_y\Big) = \sum_{x \leq s \wedge t} \delta'_x = \theta\Big(\sum_{x \leq s \wedge t} \delta_x\Big) = \theta(s \wedge t). \quad \square$$

Weisner's Theorem

Theorem (Weisner's Theorem)

Let L be a finite lattice with at least two elements, and let $a \in L$ such that $a \neq \hat{1}$. Then

$$\sum_{t\colon\, t \wedge a = \hat{0}} \mu(t, \hat{1}) = 0.$$

Sketch of Proof: Note that

$$\theta(a\delta_{\hat{1}}) = \Big(\sum_{x \le a} \delta'_x\Big)\delta'_{\hat{1}} = 0,$$

which implies that $a\delta_{\hat{1}} = 0$. On the other hand,

$$a\delta_{\hat{1}} = a \sum_{t \in L} \mu(t, \hat{1}) t = \sum_{t \in L} \mu(t, \hat{1})(a \wedge t).$$

The coefficient of $\hat{0}$ in the right hand side of the above equality, $\sum_{t \wedge a} \mu(t, \hat{1})$, must be zero because $a\delta_{\hat{1}} = 0$. □

Crosscut Theorem

Theorem (Crosscut Theorem)

Let L be a finite lattice, and let X be a subset of L such that

1. $\hat{1} \notin X$,
2. *if* $s \in L$ *and* $s \neq \hat{1}$*, then* $s \leq t$ *for some* $t \in X$.

Then

$$\mu(\hat{0}, \hat{1}) = \sum_m (-1)^m N_m,$$

where N_m *is the number of* m*-subsets of* X *whose meet is* $\hat{0}$.

Sketch of Proof: Since $\hat{1} - t = \sum_{s \not\leq t} \delta_s$, we have

$$\prod_{t \in X} (\hat{1} - t) = \sum_s \delta_s,$$

where s runs over all the elements of L satisfying $s \not\leq t$ for all $t \in X$. Then $\prod_{t \in X} (\hat{1} - t) = \delta_{\hat{1}}$ and, by expanding and equaling the coefficients of $\hat{0}$, the proof follows. □

Möbius Function of Distributive Lattices

Remark: Note that $X \subseteq L$ satisfies conditions (1) and (2) in the previous theorem iff X contains the set of coatoms of L.

Theorem

If L is a finite lattice where $\hat{0}$ is not a meet of coatoms, then $\mu(\hat{0}, \hat{1}) = 0$. Dually, if $\hat{1}$ is not a join of atoms, then $\mu(\hat{0}, \hat{1}) = 0$.

Sketch of Proof: Let $X \subset L$ such that $\hat{1} \notin X$. Then the conditions

- all $s \in L \setminus \{\hat{1}\}$ there is $x \in X$ such that $s \leq x$ and
- X contains all coatoms of L

are equivalent. Therefore the set X of coatoms of L satisfies the hypothesis of the Crosscut Theorem. Since $\hat{0}$ is not a meet of coatoms, $N_m = 0$ for each m in the Crosscut Theorem. Hence $\mu(\hat{0}, \hat{1}) = 0$. □

Möbius Function of Distributive Lattices

Theorem

Let $L = J(P)$ be a finite distributive lattice. If $[I, I'] \in Int(L)$, we have $\mu(I, I') = (-1)^{|I' \setminus I|}$ if $I' \setminus I$ is an antichain of P, and $\mu(I, I') = 0$ otherwise.

Sketch of Proof: Let $n = |I' \setminus I|$. The interval $[I, I']$ is a sublattice of $J(P)$, which is isomorphic to B_n iff $I' \setminus I$ is an antichain of P. Therefore if $I' \setminus I$ is an antichain of P, we have $\mu(I, I') = (-1)^n$. On the other hand, if $I' \setminus I$ is not an antichain of P then there exist $a, b \in I' \setminus I$ such that $a < b$. Since atoms of $[I, I']$ are of the form $I \cup \{m\}$, where m is minimal in $I' \setminus I$, the element b is not contained in any atom of $[I, I']$. Therefore I' is not the join (union) of atoms of $[I, I']$. Hence, $\mu(I, I') = 0$ when $I' \setminus I$ is not an antichain of P. □

References

- M. Bona. *A Walk Through Combinatorics*. World Scientific, New Jersey, 2012.
- G. Grätzer. *Lattice Theory*. Dover Publication, Inc., New Jersey, 2009.
- R. Stanley. *Enumerative Combinatorics*. Cambridge University Press, New York, 2012.
- J. H. van Lint and R. M. Wilson. *A Course in Combinatorics*. Cambridge University Press, Cambridge, 1994.