Terminology of Posets

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Partially Ordered Sets

Definition (Posets)

A pair $(S, \leq)$ is called *partially ordered set* (or *poset*) if S is a set and $\leq$ is a binary relation on S satisfying the following conditions:

1. $s \leq s$ for all $s \in S$ (reflexivity);
2. for all $s, r \in S$, if $s \leq r$ and $s \leq r$ then $s = r$ (antisymmetry);
3. for all r, s, t, if $s \leq r$ and $r \leq t$ then $s \leq t$ (transitivity).

We say that a poset P is a *chain* or a *totally* (or *linearly*) ordered set if for all $r, s \in P$ either $r \leq s$ or $s \leq r$.

Few more definitions:

- Let P be a poset and $r, s \in P$. If $r \leq s$ or $s \leq r$, we say that r and s are *comparable*; we write $r || s$ if r and s are not comparable.
- We use the symbols $\geq, <$, and $>$ in the obvious way; for example, $r > s$ if $s \leq r$ but $s \neq r$.

Bounds, Minimal, and Maximal Elements

Definition

Let P be a poset.

- An *upper bound* (resp., *lower bound*) of a subset S of P is an element $b \in P$ such that $s \leq b$ (resp., $s \geq b$) for all $s \in S$.
- If a lower bound of P as a subset of P exists, it must be unique; we denote it by $\hat{0}$. Dually, if a global upper bound exists it must be also unique, and we denote it by $\hat{1}$.
- An element m of P is *minimal* (resp., *maximal*) if $s \leq m$ (resp., $m \leq s$) for some $s \in P$ implies that $s = m$.

Posets (Examples)

Examples of posets:

1. For a set S the power set $\mathcal{P}(S)$ is a poset with respect to inclusion. The sets $\emptyset$ and S are the $\hat{0}$ and $\hat{1}$, respectively.
2. For $n \in \mathbb{N}$, the set D_n of all positive divisors of n is a poset if we define $d_1 \le d_2$ if d_1 divides d_2. Notice that 1 and n are the respective $\hat{0}$ and $\hat{1}$ of D_n.
3. For a set S, consider the set $\prod_S$ of partitions of S. If for $\sigma, \lambda \in \prod_S$ we define $\sigma \le \lambda$ if every block of σ is contained in a block of λ, i.e., $\sigma \le \lambda$ if σ is a refinement of λ, then $\prod_S$ is a poset. Notice that $\hat{0} = \{\{s\} \mid s \in S\}$ and $\hat{1} = \{S\}$.
4. Note that $(0, 1)$ is a poset under the standard binary relation $\le$. However, $(0, 1)$ does not contain neither $\hat{0}$ nor $\hat{1}$.

Subchains and Intervals

Definition

Let P be a poset.

- $S \subseteq P$ is a *subchain* if S is a chain by itself.
- A subchain S of P is *maximal* if it is not properly contained in any other subchain of P.
- A subchain S of P is *saturated* if $r \leq x \leq s$ for $r, s \in S$ and $x \in L$ implies that $x = r$ or $x = s$.
- The *length* $\ell(S)$ of a finite subchain S of P is $|S| - 1$.
- For $r, s \in P$ such that $r \leq s$ we define the *interval* $[r, s]$ to be the set
$$\{u \in P \mid r \leq u \leq s\}.$$
- The *length* of a finite interval $[r, s]$ of P is
$$\ell(r, s) := \max\{\ell(S) \mid S \text{ is a maximal subchain of } [r, s]\}.$$

Graded Posets

Definition

- The poset P is *graded* of *rank* n if $\ell(S) = n$ for each maximal subchain S of P.
- For $r, s \in P$, we say that r *covers* s if $s \leq r$ and $s \leq t \leq r$ implies $t \in \{r, s\}$.

Theorem

If P is a graded poset of rank n, there exists a unique rank function $\rho\colon P \to \{0, \dots, n\}$ such that $\rho(m) = 0$ if m is minimal and $\rho(r) = \rho(s) + 1$ if r covers s.

Proof: Exercise.

New Posets from Old

Let L and M be two posets.

- The *dual* of L is the pair $L^* = (L, \leq_d)$, where $r \leq_d s$ iff $s \leq r$ in L.
- The *disjoint union* of L and M is the pair $L + M = (L \cup M, \leq_{du})$, where $r \leq_{du} s$ iff $r, s \in L$ and $r \leq s$ in L, or $r, s \in M$ and $r \leq s$ in M.
- The *ordinal sum* of L and M is the pair $L \oplus M = (L \cup M, \leq_{os})$, where $r \leq_{os} s$ iff (a) $r, s \in L$ and $r \leq s$ in L, (b) $r, s \in M$ and $r \leq s$ in M, or (c) $r \in L$ and $s \in M$.
- The *direct product* of L and M is the pair $L \times M = (L \times M, \leq_{dp})$, where $(r, s) \leq_{dp} (r', s')$ iff $r \leq r'$ in L and $s \leq s'$ in M.

Theorem

If L and M are posets, so are $L^*, L + M, L \oplus M$, and $L \times M$.

Morphisms of Posets

Definition

Let $\varphi\colon R \to S$ be a map between posets.

- φ is called *order-preserving* (resp.,*order-reflecting* if for all $r, r' \in R$ such that $r \le r'$ we have $\varphi(r) \le \varphi(r')$ (resp., $\varphi(r) \ge \varphi(r')$).
- φ is an *order-embedding* if it is both order-preserving and order-reflecting.
- φ is an *isomorphism* of posets if it is a bijective order-embedding; in this case R and S are said to be *isomorphic*.

Remarks:

- An order-embedding is injective.
- Two posets R and S are isomorphic iff there are order-preserving maps $\varphi\colon R \to S$ and $\psi\colon S \to R$ such that $\varphi \circ \psi = \mathrm{Id}_S$ and $\psi \circ \varphi = \mathrm{Id}_R$.

References

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R. Stanley. *Enumerative Combinatorics*. Cambridge University Press, New York, 2012.