# IDEAL THEORY AND PRÜFER DOMAINS

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## Lecture 0: Localization

**Localization of Rings.** Let $R$ be a commutative ring with identity, and let $S$ be a submonoid of $(R \setminus \{0\}, \cdot)$. Now one can define the following relation on $R \times S$: $(r_1, s_1) \sim (r_2, s_2)$ for $(r_1, s_1), (r_2, s_2) \in R \times S$ provided that $(r_1 s_2 - r_2 s_1)s = 0$ for some $s \in S$. It is not hard to check that $\sim$ is indeed an equivalence relation on $R \times S$. We let $S^{-1}R$ denote the set of equivalence classes of $\sim$ and, for $r \in R$ and $s \in S$, we let $r/s$ denote the equivalence class of $(r, s)$. Motivated by the standard addition and multiplication of rational numbers, we can now define for $r_1/s_1$ and $r_2/s_2$ in $S^{-1}R$ the following operations:

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} := \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} := \frac{r_1 r_2}{s_1 s_2}.$$

It is routine to verify that both operations are well defined and that $(S^{-1}R, +, \cdot)$ is a commutative ring with identity $1/1$.

**Proposition 1.** *$(S^{-1}R, +, \cdot)$ is a commutative ring with identity.*

The ring $S^{-1}R$ is called the *localization* of $R$ at $S$. We can easily see that the map $\pi \colon R \to S^{-1}R$ defined by $\pi(r) = r/1$ satisfies the properties in the following proposition.

**Proposition 2.** *Let $R$ be a commutative ring with identity, and let $S$ be a submonoid of $(R \setminus \{0\}, \cdot)$. Then the following statements hold.*

(1) *The map $\pi \colon R \to S^{-1}R$ is a ring homomorphism satisfying that $\pi(s)$ is a unit in $S^{-1}R$ for every $s \in S$. In addition, $\pi$ is injective if and only if $S$ contains no zero-divisors of $R$.*

(2) *If $\varphi \colon R \to T$ is a ring homomorphism such that $\varphi(s)$ is a unit in $T$ for every $s \in S$, then there exists a unique ring homomorphism $\theta \colon S^{-1}R \to T$ such that $\varphi = \theta \circ \pi$.*

*Proof.* (1) One can readily see that $\pi$ is a ring homomorphism. For every $s \in S$, it is clear that $1/s \in S^{-1}R$ and, therefore, $\pi(s) = s/1$ is a unit in $S^{-1}R$. If $s \in S$ is a zero-divisor in $R$, then taking $r \in R \setminus \{0\}$ with $sr = 0$, we can see that $\pi(r) = 0$ and so $\pi$ is not injective. Conversely, if $\pi(r) = 0$ for some $r \in R \setminus \{0\}$, then $r/1 = 0/1$ and so there is an $s \in S$ such that $sr = 0$.

(2) For $\varphi$ as in (2), define $\theta\colon S^{-1}R \to T$ by $\theta(r/s) = \varphi(r)\varphi(s)^{-1}$. Since $\varphi(s) \in T^\times$ for every $s \in S$, the element $\varphi(r)\varphi(s)^{-1}$ belongs to $T$, and it is easy to check that $\theta$ is a well-defined ring homomorphism. Since $\theta(\pi(r)) = \theta(r/1) = \varphi(r)$, the equality $\theta \circ \pi = \varphi$ holds. Finally, for any ring homomorphism $\theta'\colon S^{-1}R \to T$ with $\varphi = \theta' \circ \pi$, we see that $\theta'(r/s) = \theta'(r/1)\theta'(1/s) = \theta'(\pi(r))\theta'(\pi(s))^{-1} = \varphi(r)\varphi(s)^{-1} = \theta(r/s)$ for all $r/s \in S^{-1}R$. Hence $\theta' = \theta$, and the uniqueness follows. □

For an ideal $I$ of $R$, the ideal $S^{-1}R\,\pi(I)$ of $S^{-1}R$ is called the *extension* of $I$ by $\pi$ and is denoted by $S^{-1}I$. Observe that every element of $S^{-1}I$ can be written as $a/s$ for some $a \in I$ and $s \in S$.

**Proposition 3.** *Let $R$ be a commutative ring with identity, and let $S$ be a submonoid of $(R \setminus \{0\}, \cdot)$. Then the following statements hold.*

1. *For any ideal $J$ of $S^{-1}R$ the equality $S^{-1}\pi^{-1}(J) = J$ holds. In particular, every ideal of $S^{-1}R$ is the extension of an ideal in $R$.*
2. *For an ideal $I$ of $R$, the equality $S^{-1}I = S^{-1}R$ holds if and only if $I \cap S \neq \emptyset$.*
3. *The assignment $I \mapsto S^{-1}I$ induces a bijection between the set of prime ideals of $R$ disjoint from $S$ and the set of prime ideals of $S^{-1}R$.*

*Proof.* (1) It suffices to show that $J$ is contained in the ideal $J' := S^{-1}\pi^{-1}(J)$. Take $r/s \in J$. As $r/1 = (s/1)(r/s) \in J$, it follows that $r \in \pi^{-1}(J)$, and so $r/1 \in S^{-1}\pi^{-1}(J)$. Since $J'$ is an ideal of $S^{-1}R$, we see that $r/s = (1/s)(r/1) \in J'$. Hence $J' = J$. The second statement is an immediate consequence of the first one.

(2) If $S^{-1}I = S^{-1}R$, then $a/s = 1/1$ for some $a \in I$ and $s \in S$. So we can take $s' \in S$ such that $(a - s)s' = 0$. This means that $ss' = as' \in I$, whence $I \cap S = \emptyset$. Conversely, assume that $I \cap S \neq \emptyset$ and take $a \in I \cap S$. Then for all $r/s \in S^{-1}R$, we see that $ra \in I$ while $sa \in S$, which implies that $r/s = (ra)/(sa) \in S^{-1}I$. Thus, $S^{-1}I = S^{-1}R$.

(3) Let $\mathscr{I}$ be the set of prime ideals in $R$ that are disjoint from $S$, and let $\mathscr{J}$ be the set of prime ideals in $S^{-1}R$. Let $e\colon \mathscr{I} \to \mathscr{J}$ and $c\colon \mathscr{J} \to \mathscr{I}$ be the maps given by the assignments $I \mapsto S^{-1}I$ and $J \mapsto \pi^{-1}(J)$, respectively. Since homomorphic inverse images of prime ideals are prime ideals, $c$ is well defined. To check that $e$ is also well defined, take $P \in \mathscr{I}$ and let us verify that $S^{-1}P$ is a prime ideal. Take $r_1, r_2 \in R$ and $s_1, s_2 \in S$ such that $(r_1/s_1)(r_2/s_2) \in S^{-1}P$. Then there are elements $a \in P$ and $s, s' \in S$ such that $(r_1r_2s - as_1s_2)s' = 0$, which implies that $r_1r_2ss' \in P$. As $P$ is prime and disjoint from $S$, we obtain that either $r_1 \in P$ or $r_2 \in P$, from which we deduce that either $r_1/s_1 \in S^{-1}P$ or $r_2/s_2 \in S^{-1}P$. Hence $S^{-1}P$ is a prime ideal, and so the map $e$ is well defined. Part (1) guarantees that $e \circ c$ is the identity of $\mathscr{J}$. Proving that $c \circ e$ is the identity of $\mathscr{I}$ amounts to arguing that $c(e(P)) \subseteq P$ for every $P \in \mathscr{I}$. To do so, take $a_3/s_3 \in e(P) = S^{-1}P$ for $a_3 \in P$ and $s_3 \in S$. If $r \in \pi^{-1}(a_3/s_3)$, then $r/1 = a_3/s_3$ and there is an $s'' \in S$ with $(rs_3 - a_3)s'' = 0$. This implies that $rs_3 \in P$, from which we deduce that $r \in P$. Hence $c(e(P)) \subseteq P$, as desired. Thus, $c \circ e$ is the identity of $\mathscr{I}$, which completes the proof. □

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