IDEAL THEORY AND PRÜFER DOMAINS

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Localization

Localization of Rings. Let R be a commutative ring with identity. A *multiplicative subset* of R is a submonoid of $(R \setminus \{0\}, \cdot)$. Let S be a multiplicative subset of R. One can define the following relation on $R \times S$: $(r_1, s_1) \sim (r_2, s_2)$ for $(r_1, s_1), (r_2, s_2) \in R \times S$ provided that $(r_1 s_2 - r_2 s_1)s = 0$ for some $s \in S$. It is not hard to check that $\sim$ is indeed an equivalence relation on $R \times S$. We let $S^{-1}R$ denote the set of equivalence classes of $\sim$ and, for $r \in R$ and $s \in S$, we let r/s denote the equivalence class of (r, s). Motivated by the standard addition and multiplication of rational numbers, we can now define for r_1/s_1 and r_2/s_2 in $S^{-1}R$ the following operations:

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} := \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} := \frac{r_1 r_2}{s_1 s_2}.$$

It is routine to verify that both operations are well defined and that $(S^{-1}R, +, \cdot)$ is a commutative ring with identity $1/1$.

Proposition 1. *$(S^{-1}R, +, \cdot)$ is a commutative ring with identity.*

The ring $S^{-1}R$ is called the *localization* of R at S. We can easily see that the map $\pi \colon R \to S^{-1}R$ defined by $\pi(r) = r/1$ satisfies the properties in the following proposition.

Proposition 2. *Let R be a commutative ring with identity, and let S be a multiplicative subset of R. Then the following statements hold.*

(1) *The map $\pi \colon R \to S^{-1}R$ is a ring homomorphism satisfying that $\pi(s)$ is a unit in $S^{-1}R$ for every $s \in S$. In addition, π is injective if and only if S contains no zero-divisors of R.*

(2) *If $\varphi \colon R \to T$ is a ring homomorphism such that $\varphi(s)$ is a unit in T for every $s \in S$, then there exists a unique ring homomorphism $\theta \colon S^{-1}R \to T$ such that $\varphi = \theta \circ \pi$.*

Proof. (1) One can readily see that π is a ring homomorphism. For every $s \in S$, it is clear that $1/s \in S^{-1}R$ and, therefore, $\pi(s) = s/1$ is a unit in $S^{-1}R$. If $s \in S$ is a zero-divisor in R, then taking $r \in R \setminus \{0\}$ with $sr = 0$, we can see that $\pi(r) = 0$ and so π is not injective. Conversely, if $\pi(r) = 0$ for some $r \in R \setminus \{0\}$, then $r/1 = 0/1$ and so there is an $s \in S$ such that $sr = 0$.

(2) For φ as in (2), define $\theta\colon S^{-1}R \to T$ by $\theta(r/s) = \varphi(r)\varphi(s)^{-1}$. Since $\varphi(s) \in T^\times$ for every $s \in S$, the element $\varphi(r)\varphi(s)^{-1}$ belongs to T, and it is easy to check that θ is a well-defined ring homomorphism. Since $\theta(\pi(r)) = \theta(r/1) = \varphi(r)$, the equality $\theta \circ \pi = \varphi$ holds. Finally, for any ring homomorphism $\theta'\colon S^{-1}R \to T$ with $\varphi = \theta' \circ \pi$, we see that $\theta'(r/s) = \theta'(r/1)\theta'(1/s) = \theta'(\pi(r))\theta'(\pi(s))^{-1} = \varphi(r)\varphi(s)^{-1} = \theta(r/s)$ for all $r/s \in S^{-1}R$. Hence $\theta' = \theta$, and the uniqueness follows. □

If R is an integral domain, then we can take S to be $(R\setminus\{0\}, \cdot)$, then the localization of R at S is clearly a field. In this case, $S^{-1}R$ is called the *quotient field* or the *field of fractions* of R and is denoted by $\mathrm{qf}(R)$. Note that $\mathbb{Q}$ is the quotient field of $\mathbb{Z}$. The following two examples of localizations show often in commutative ring theory.

Example 3. Let R be a commutative ring with identity, and let P be a prime ideal of R. Since R is prime, $S := R \setminus P$ is a multiplicative subset of R. The ring $S^{-1}R$ is called the *localization of R at P* and is denoted by R_P.

(1) For instance, if $p \in \mathbb{P}$, then
$$\mathbb{Z}_{(p)} = \big\{m/n : m, n \in \mathbb{Z} \text{ and } p \nmid n\big\};$$
observe that the units of $\mathbb{Z}_{(p)}$ are the elements m/n such that $m, n \in \mathbb{Z}$ and $p \nmid mn$.

(2) Set $R = \mathbb{C}[x, y]$ and $P = (x, y)$. Then P is a prime ideal, and the localization R_P of R at P consists of all rational expressions f/g, where $f, g \in R$ and $g \notin P$, that is, $g(0, 0) \neq 0$. The units of R_P are the rational expressions f/g satisfying $f(0, 0)g(0, 0) \neq 0$.

In general, the units of R_P have the form r/s with $r, s \in R$ such that $rs \notin P$.

Example 4. Let R be a commutative ring with identity, and let f be an element of R such that $f^n \neq 0$ for any $n \in \mathbb{N}_0$. For $S := \{f^n : n \in \mathbb{N}_0\}$, the ring $S^{-1}R = R[1/f]$ is often denoted by R_f. It is not hard to argue that R_f is isomorphic to the ring $R[x]/(xf-1)$. For instance, $\mathbb{Z}[x]_x = \mathbb{Z}[x, 1/x]$, which is the ring of Laurent polynomials in one variable over $\mathbb{Z}$.

An integral domain is the intersection of all its localizations at prime ideals.

Proposition 5. *If R is an integral domain, then $R = \bigcap_P R_P = \bigcap_M R_M$, where the first intersection runs over all prime ideals of R and the second intersection runs over all maximal ideals of R.*

Proof. It is clear that $R \subseteq \bigcap_P R_P \subseteq \bigcap_M R_M$. To show that $\bigcap_M R_M \subseteq R$, take $a \in \bigcap_M R_M$ and suppose, by way of contradiction, that $a \notin R$. The set $I_a := \{r \in R : ra \in R\}$ is an ideal of R, which is a proper ideal because $a \notin R$. Let M be a maximal ideal of R containing I_a. Then $a \in R_M$, and we can take $r \in R$ and $s \in R \setminus M$ such that $a = r/s$. As $sa = r \in R$, we see that $s \in I_a \subseteq M$, which is a contradiction. □

Localization and Ideals. For an ideal I of R, the ideal $S^{-1}R\,\pi(I)$ of $S^{-1}R$ is called the *extension* of I by π and is denoted by $S^{-1}I$. Observe that every element of $S^{-1}I$ can be written as a/s for some $a \in I$ and $s \in S$.

Proposition 6. *Let R be a commutative ring with identity, and let S be a multiplicative subset of R. Then the following statements hold.*

(1) *For any ideal J of $S^{-1}R$ the equality $S^{-1}\pi^{-1}(J) = J$ holds. In particular, every ideal of $S^{-1}R$ is the extension of an ideal in R.*

(2) *For an ideal I of R, the equality $S^{-1}I = S^{-1}R$ holds if and only if $I \cap S \neq \emptyset$.*

(3) *The assignment $I \mapsto S^{-1}I$ induces a bijection between the set of prime ideals of R disjoint from S and the set of prime ideals of $S^{-1}R$.*

Proof. (1) It suffices to show that J is contained in the ideal $J' := S^{-1}\pi^{-1}(J)$. Take $r/s \in J$. As $r/1 = (s/1)(r/s) \in J$, it follows that $r \in \pi^{-1}(J)$, and so $r/1 \in S^{-1}\pi^{-1}(J)$. Since J' is an ideal of $S^{-1}R$, we see that $r/s = (1/s)(r/1) \in J'$. Hence $J' = J$. The second statement is an immediate consequence of the first one.

(2) If $S^{-1}I = S^{-1}R$, then $a/s = 1/1$ for some $a \in I$ and $s \in S$. So we can take $s' \in S$ such that $(a - s)s' = 0$. This means that $ss' = as' \in I$, whence $I \cap S = \emptyset$. Conversely, assume that $I \cap S \neq \emptyset$ and take $a \in I \cap S$. Then for all $r/s \in S^{-1}R$, we see that $ra \in I$ while $sa \in S$, which implies that $r/s = (ra)/(sa) \in S^{-1}I$. Thus, $S^{-1}I = S^{-1}R$.

(3) Let $\mathscr{I}$ be the set of prime ideals in R that are disjoint from S, and let $\mathscr{J}$ be the set of prime ideals in $S^{-1}R$. Let $e\colon \mathscr{I} \to \mathscr{J}$ and $c\colon \mathscr{J} \to \mathscr{I}$ be the maps given by the assignments $I \mapsto S^{-1}I$ and $J \mapsto \pi^{-1}(J)$, respectively. Since homomorphic inverse images of prime ideals are prime ideals, c is well defined. To check that e is also well defined, take $P \in \mathscr{I}$ and let us verify that $S^{-1}P$ is a prime ideal. Take $r_1, r_2 \in R$ and $s_1, s_2 \in S$ such that $(r_1/s_1)(r_2/s_2) \in S^{-1}P$. Then there are elements $a \in P$ and $s, s' \in S$ such that $(r_1r_2s - as_1s_2)s' = 0$, which implies that $r_1r_2ss' \in P$. As P is prime and disjoint from S, we obtain that either $r_1 \in P$ or $r_2 \in P$, from which we deduce that either $r_1/s_1 \in S^{-1}P$ or $r_2/s_2 \in S^{-1}P$. Hence $S^{-1}P$ is a prime ideal, and so the map e is well defined. Part (1) guarantees that $e \circ c$ is the identity of $\mathscr{J}$. Proving that $c \circ e$ is the identity of $\mathscr{I}$ amounts to arguing that $c(e(P)) \subseteq P$ for every $P \in \mathscr{I}$. To do so, take $a_3/s_3 \in e(P) = S^{-1}P$ for $a_3 \in P$ and $s_3 \in S$. If $r \in \pi^{-1}(a_3/s_3)$, then $r/1 = a_3/s_3$ and there is an $s'' \in S$ with $(rs_3 - a_3)s'' = 0$. This implies that $rs_3 \in P$, from which we deduce that $r \in P$. Hence $c(e(P)) \subseteq P$, as desired. Thus, $c \circ e$ is the identity of $\mathscr{I}$, which completes the proof. □

The property of being Noetherian is preserved under localization.

Proposition 7. *Let R be a Noetherian domain, and let S be a multiplicative subset of R. Then $S^{-1}R$ is also Noetherian.*

Proof. By Proposition 6, any ideal of $S^{-1}R$ has the form $S^{-1}I$ for some ideal I of R. Since R is Noetherian, $I = Ra_1 + \cdots + Ra_n$ for some $a_1, \dots, a_n \in R$. Then for each $a/s \in S^{-1}I$ with $a \in I$ and $s \in S$, we can write $a = \sum_{i=1}^n r_i a_i$ for some $r_1, \dots, r_n \in R$ to obtain the equality $a/s = \sum_{i=1}^n (r_i/s)(a_i/1)$. Thus, $S^{-1}I$ is the ideal of $S^{-1}R$ generated by $a_1/1, \dots, a_n/1$. Hence $S^{-1}R$ is a Noetherian ring. □

In addition, localization preserves the most important ideal operations, as we will see in the following proposition.

Proposition 8. *Let R be a commutative ring with identity, and let S be a multiplicative subset of R. For ideals I and J of R, the following statements hold.*

(1) $S^{-1}(I + J) = S^{-1}I + S^{-1}J$.
(2) $S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J$.
(3) $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$.
(4) $S^{-1}R \,/\, S^{-1}I \cong S^{-1}(R/I)$.

Proof. Exercise. □

Localization of Modules. We can localize modules in the same way we have localized rings. Let R be a commutative ring with identity with a multiplicative subset S, and let M be an R-module. It is easy to verify that the relation on $M \times S$ defined by $(m_1, s_1) \sim (m_2, s_2)$ if there is an $s \in S$ such that $(m_1 s_2 - m_2 s_1)s = 0$ is an equivalence relation, and one denotes the class of (m, s) by m/s and the set of all equivalence classes by $S^{-1}M$. It is routine to verify that the operations

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} := \frac{s_2 m_1 + s_1 m_2}{s_1 s_2} \quad \text{and} \quad \frac{r}{s} \cdot \frac{m_1}{s_1} := \frac{r m_1}{s s_1},$$

where $m_1/s_1, m_2/s_2 \in S^{-1}M$ and $r/s \in S^{-1}R$, are well defined and turn $S^{-1}M$ into an $S^{-1}R$-module, which is called the *localization* M at S. In particular, $S^{-1}M$ is an R-module. As Exercise 7 indicates, localization commutes with (direct) sums, intersections, and quotients of modules. The map $\pi \colon M \to S^{-1}M$ defined by $m \mapsto m/1$ is an R-module homomorphism and has the universal property described in Proposition 9(2).

Proposition 9. *Let R be a commutative ring with identity, let S be a multiplicative subset of R, and let M be an R-module. Then the following statements hold.*

(1) *The map $\pi \colon M \to S^{-1}M$ defined by $m \mapsto m/1$ is an R-module homomorphism and $\ker \pi = \{m \in M : sm = 0 \text{ for some } s \in S\}$.*
(2) *If M' is an R-module such that, for each $s \in S$, left multiplication by s yields a bijection on M' and, in addition, $\varphi \colon M \to M'$ is an R-module homomorphism, then there is a unique R-module homomorphism $\theta \colon S^{-1}M \to M'$ such that $\varphi = \theta \circ \pi$.*

(3) *Any R-module homomorphism $\psi\colon M \to M'$ induces an $S^{-1}R$-module homomorphism $S^{-1}M \to S^{-1}M'$ via the assignment $m/s \mapsto \psi(m)/s$.*

Proof. Exercise. □

The localization of a Noetherian R-module is Noetherian.

Proposition 10. *Let R be a commutative ring with identity, and let S be a multiplicative subset of R. If M is a Noetherian R-module, then $S^{-1}M$ is also a Noetherian $S^{-1}R$-module.*

Proof. See the proof of Proposition 7. □

Exercises

Exercise 1. *Let R be a commutative ring with identity, and let S be a multiplicative subset of R. The set $\bar{S} := \{r \in R : \pi(r)$ is a unit of $S^{-1}R\}$ is called the* saturation *of S. Prove the following statements.*

(1) $\bar{S} = \{r \in R : rt \in S$ *for some* $t \in R\}$.

(2) $\bar{S}$ *is a multiplicative subset of R satisfying $S \subseteq \bar{S} = \bar{\bar{S}}$.*

(3) $S^{-1}R \cong \bar{S}^{-1}R$.

Exercise 2. *Let R be a commutative ring with identity, and let I and J be ideals of R. Prove that $I = J$ if and only if $IR_P = JR_P$ for every maximal ideal P of R.*

Exercise 3. *Let R be an integral domain, and let S be a multiplicative subset of R. Prove the following statements.*

(1) *If R is a UFD, then $S^{-1}R$ is a UFD.*

(2) *Suppose that S is saturated and R is atomic (i.e, every nonzero nonunit of R factors into irreducibles). If $S^{-1}R$ is a UFD, then R is a UFD.*

Exercise 4. *Prove Proposition 8.*

Exercise 5. *Prove Proposition 9.*

Exercise 6. *Let R be a commutative ring, and let S be a multiplicative subset of R. Let M be an R-module. Let $\pi\colon M \to S^{-1}M$ be the natural map. Prove the following statements.*

(1) *For each R-submodule N of M, the set $S^{-1}N := \{n/s : n \in N$ and $s \in S\}$ is an $S^{-1}R$-submodule of $S^{-1}M$.*

(2) *If L is an $S^{-1}R$-submodule of $S^{-1}M$, then $\pi^{-1}(L)$ is an R-submodule of M.*

(3) *If N is an R-submodule of M, then $N \subseteq \pi^{-1}(S^{-1}N)$. Also, if $N = \pi^{-1}(L)$ for an $S^{-1}R$-submodule L of $S^{-1}M$, then $L = S^{-1}N$. In particular, every $S^{-1}R$-submodule of $S^{-1}M$ has the form $S^{-1}N$ for an R-submodule N of M.*

(4) *Deduce that there is a bijection between the set of $S^{-1}R$-submodules of $S^{-1}M$ and the set of R-submodules N of M satisfying the condition: if $sm \in N$ for some $s \in S$ and $m \in M$, then $m \in N$.*

Exercise 7. *Let R be a commutative ring with identity, let S be a multiplicative subset of R, and let M be an R-module. For any submodules M_1 and M_2 of M, prove the following statements.*

(1) $S^{-1}(M_1 + M_2) = S^{-1}M_1 + S^{-1}M_2$.
(2) $S^{-1}(M_1 \oplus M_2) = S^{-1}M_1 \oplus S^{-1}M_2$.
(3) $S^{-1}(M_1 \cap M_2) \cong S^{-1}M_1 \cap S^{-1}M_2$.
(4) $S^{-1}M \,/\, S^{-1}M_1 = S^{-1}(M/M_1)$.

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