IDEAL THEORY AND PRÜFER DOMAINS

FELIX GOTTI

Lecture 2: Noetherian Rings

Basic Properties. Let R be a commutative ring. Recall that R is Noetherian if every ideal of R is finitely generated. We can characterize Noetherian rings as follows.

Proposition 1. *For a commutative ring R, the following statements are equivalent.*

(a) *R is Noetherian.*

(b) *R satisfies the ascending chain condition (ACC) on ideals: every ascending chain of ideals of R eventually stabilizes.*

(c) *Every nonempty set of ideals of R contains a maximal element (under inclusion).*

Proof. Exercise. □

Example 2. PIDs and, in particular, Euclidean domains are Noetherian rings. In addition, the rings of integers of algebraic number fields are Noetherian, even though many of them are not PIDs. On the other hand, not every UFD is Noetherian; for instance, $\mathbb{Z}[x_1, x_2, \dots]$ is a UFD but its prime ideal $(x_1, x_2, \dots)$ is not finitely generated.

Let us briefly discuss some results that allow us to produce Noetherian rings. Quotients and localizations of Noetherian rings are Noetherian rings.

Proposition 3. *Let R be a commutative ring. For an ideal I of R and a submonoid S of $(R \setminus \{0\}, \cdot)$, the following statements hold.*

(1) *If R is Noetherian, then so is the quotient R/I.*

(2) *If R is Noetherian, then so is the localization $S^{-1}R$.*

Proof. (1) Every ideal of R/I is of the form J/I, where J is an ideal of R containing I. Fix an ideal J/I of R/I. Since R is Noetherian, we can take $r_1, \dots, r_n \in R$ such that $J = (r_1, \dots, r_n)$. Hence $J/I = (r_1 + I, \dots, r_n + I)$, and so it is a finitely generated ideal. Thus, R/I is also Noetherian.

(2) We have seen that every ideal of $S^{-1}R$ has the form $S^{-1}J$ for some ideal J of R. Let $S^{-1}J$ be an ideal of $S^{-1}R$. As R is Noetherian, $J = (r_1, \dots, r_n)$ for some $r_1, \dots, r_n \in R$. This implies that $S^{-1}J = (r_1/1, \dots, r_n/1)$. Since every ideal of $S^{-1}R$ is finitely generated, $S^{-1}R$ is Noetherian. □

Another important tool to produce Noetherian rings is Hilbert Basis Theorem. Although its proof is not straightforward, it can be found in any standard algebra textbook (for instance, see [1, page 316]).

Theorem 4 (Hilbert Basis Theorem)**.** *If R is a Noetherian ring, then $R[x]$ is also a Noetherian ring.*

The following corollary is an immediate consequence of Hilbert Basis Theorem.

Corollary 5. *If R is a Noetherian ring, then $R[x_1, \dots, x_n]$ is a Noetherian ring.*

Primary Decomposition. The main purpose of this section is to prove the Lasker-Noether Decomposition Theorem, which states that every ideal in a Noetherian ring can be expressed as an intersection of primary ideals. This result was first proved by E. Lasker in the context of polynomial rings, and the proof was then simplified and generalized by E. Noether. Throughout this subsection, R is a commutative ring with identity.

Recall that every proper ideal of R is contained in a maximal ideal. The *radical* (or *nilradical*) of a proper ideal I of R, denoted by $\operatorname{Rad} I$, is the intersection of all prime ideals of R containing I. In addition, $\operatorname{Rad} R = R$. The ideal I is *radical* if $\operatorname{Rad} I = I$. Clearly, every prime ideal is radical.

Example 6. In $\mathbb{Z}$, the radical of $18\mathbb{Z}$ is $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ and the radical of both $9\mathbb{Z}$ and $27\mathbb{Z}$ is the ideal $3\mathbb{Z}$.

Proposition 7. *Let R be a commutative ring with identity, and let $I, I_1, \dots, I_n$ be ideals of R. Then the following statements hold.*

(1) $\operatorname{Rad} I = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}$.
(2) $\operatorname{Rad}(\operatorname{Rad} I) = \operatorname{Rad} I$.
(3) $\operatorname{Rad} I_1 \cdots I_n = \operatorname{Rad}\left(\bigcap_{j=1}^n I_j\right) = \bigcap_{j=1}^n \operatorname{Rad} I_j$, *and so* $\operatorname{Rad} I^n = \operatorname{Rad} I$.

Proof. Exercise. □

A proper ideal Q of R is called *primary* if whenever $rs \in Q$ for some $r, s \in R$, the fact that $r \notin Q$ implies that $s^n \in Q$ for some $n \in \mathbb{N}$. Clearly, every prime ideal is primary. The converse does not hold even in $\mathbb{Z}$, where $4\mathbb{Z}$ is a primary ideal that is not prime. However, observe that $\operatorname{Rad} 4\mathbb{Z} = 2\mathbb{Z}$, which is a prime ideal. This is not a coincidence, as the following proposition indicates.

Proposition 8. *Let R be a commutative ring with identity, and let Q be an ideal of R. Then the following statements hold.*

(1) *If Q is primary, then $\operatorname{Rad} Q$ is prime.*
(2) *If $\operatorname{Rad} Q$ is maximal, then Q is primary.*

(3) *If M is a maximal ideal such that $M^n \subseteq Q \subseteq M$ for some $n \in \mathbb{N}$, then Q is primary and* $\operatorname{Rad} Q = M$.

Proof. Exercise. □

A proper ideal I of R is *irreducible* if for any ideals J and K in R such that $I = J \cap K$ either $J = I$ or $K = I$. Clearly, every prime ideal is irreducible.

Lemma 9. *In a Noetherian ring, every irreducible ideal is primary.*

Proof. Let R be Noetherian ring, and let Q be an irreducible ideal of R. Take $a, b \in R$ such that $ab \in Q$ but $b \notin Q$. For each $n \in \mathbb{N}$, consider the colon ideal $A_n := \{r \in R : ra^n \in Q\}$. One can readily see that $(A_n)_{n \in \mathbb{N}}$ is an ascending chain of ideals. Since R is Noetherian, there is an $n \in \mathbb{N}$ such that $A_m = A_n$ for every $m \geq n$. Now consider the ideals $I := (a^n) + Q$ and $J := (b) + Q$. It is clear that $Q \subseteq I \cap J$. To argue the reverse inclusion, take $y \in I \cap J$ and write $y = ra^n + q$ for some $r \in R$ and $q \in Q$. As $aJ \subseteq Q$, it follows that $ay \in Q$. Therefore $ra^{n+1} = ay - aq \in Q$. This implies that $r \in A_{n+1} = A_n$, and so $y = ra^n + q \in Q$. Thus, $Q = I \cap J$. Because Q is irreducible, $Q = I$ or $Q = J$. Now the fact that $b \notin Q$ ensures that $Q = I$, and so $a^n \in Q$. Hence Q is a primary ideal. □

Even in the context of Noetherian rings not every primary ideal is irreducible, as the following example shows.

Example 10. Consider the ideal $Q = (x, y)^2$ of $\mathbb{Q}[x, y]$. Since Q is a power of the maximal ideal (x, y), it must be primary. However, Q is not irreducible because it is the intersection of the ideals $(x, y^2) = (x) + (x, y)^2$ and $(y, x^2) = (y) + (x, y)^2$.

An ideal I of R has a *primary decomposition* if $I = \bigcap_{j=1}^n Q_j$ for some primary ideals $Q_1, \dots, Q_n$. Such a decomposition is called *irredundant* if the radicals of the $I_1, \dots, I_n$ are all distinct and $\bigcap_{j \neq k} Q_j \not\subseteq Q_k$ for any $k \in [\![1, n]\!]$. If every ideal in R has an irredundant primary decomposition, then R is called a *Lasker ring.* We are now in a position to prove that every Noetherian ring is a Lasker ring.

Theorem 11 (Lasker-Noether Decomposition Theorem)**.** *Every proper ideal in a Noetherian ring has an irredundant primary decomposition.*

Proof. Let R be a Noetherian ring. By virtue of Lemma 9, proving that every proper ideal of R has a primary decomposition amounts to arguing that every proper ideal of R is the finite intersection of irreducible ideals. Suppose, by way of contradiction, that this is not the case, and let $\mathscr{S}$ be the set of all the ideals of R that cannot be written as finite intersections of irreducible ideals. Clearly, $\mathscr{S}$ is a nonempty poset, and it is routine to verify that every chain in $\mathscr{S}$ has an upper bound. By Zorn's Lemma, there exists a maximal element J in $\mathscr{S}$. Since J belongs to $\mathscr{S}$, it is not irreducible and so there exist ideals I_1 and I_2 both properly containing J such that $J = I_1 \cap I_2$. The maximality of J now implies that both I_1 and I_2 can be written as finite intersections of irreducible ideals in R. However, this immediately implies that J can be also written as a finite intersection of irreducible ideals in R, contradicting that J belongs to $\mathscr{S}$. Thus, every proper ideal of R has a primary decomposition. We leave to the reader to show that we can make any primary decomposition irredundant. $\square$

References

[1] D. S. Dummit and R. M. Foote: *Abstract Algebra* (Third Edition), John Wiley & Sons, 2004.

Department of Mathematics, MIT, Cambridge, MA 02139
Email address: fgotti@mit.edu