IDEAL THEORY IN PRÜFER DOMAINS

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Lecture 3: Krull Intersection Theorem

Krull Intersection Theorem. The following proposition is a version of Krull Intersection Theorem for Noetherian rings with potential zero divisors. The proof that we discuss does not use primary decomposition, and was given by H. Perdry in [1].

Proposition 1. *Let R be a commutative ring that is Noetherian, and let I be an ideal of R. Then there exists $r \in I$ such that $(1-r)\bigcap_{n\in\mathbb{N}} I^n = (0)$.*

Proof. Write $I = (a_1, \dots, a_\ell)$ and $\bigcap_{n\in\mathbb{N}} I^n = (b_1, \dots, b_k)$. Now fix $j \in [\![1, k]\!]$. For every $n \in \mathbb{N}$, the fact that $b_j \in I^n$ guarantees the existence of a homogeneous polynomial $p_n \in R[x_1, \dots, x_\ell]$ such that $b_j = p_n(a_1, \dots, a_\ell)$. For each $n \in \mathbb{N}$, consider the ideal $J_n = (p_1, \dots, p_n)$ of $R[x_1, \dots, x_\ell]$. Since the chain of ideals $(J_n)_{n\in\mathbb{N}}$ is ascending and $R[x_1, \dots, x_\ell]$ is a Noetherian ring by Hilbert Basis Theorem, there is an $n \in \mathbb{N}$ such that $J_{n+1} = J_n$. In particular, p_{n+1} belongs to J_n. As a result, we can take polynomials $q_1, \dots, q_n \in R[x_1, \dots, x_\ell]$ such that $p_{n+1} = \sum_{i=1}^n q_i p_{n+1-i}$. Observe that there is no loss of generality in assuming that q_d is a homogeneous polynomial of degree d for every $d \in [\![1, n]\!]$, and we do so. After evaluating both sides of $p_{n+1} = \sum_{i=1}^n q_i p_{n+1-i}$ at $(x_1, \dots, x_\ell) = (a_1, \dots, a_\ell)$, we see that

$$b_j = (q_1(a_1, \dots, a_\ell) + \cdots + q_{n+1}(a_1, \dots, a_\ell))b_j = r_j b_j$$

for some $r_j \in I$. Therefore, for every $j \in [\![1, k]\!]$, we have found $r_j \in I$ satisfying that $(1-r_j)b_j = 0$. Then the product $(1-r_1)\cdots(1-r_k)$ annihilates b_j for every $j \in [\![1, k]\!]$. Hence $(1-r)\bigcap_{n\in\mathbb{N}} I^n = (0)$ when $r = 1-(1-r_1)\cdots(1-r_k)$. □

The previous proposition is specially useful in the context of integral domains and local rings.

Theorem 2 (Krull Intersection Theorem)**.** *Let R be a Noetherian domain or a Noetherian local ring, and let I be a proper ideal of R. Then $\bigcap_{n\in\mathbb{N}} I^n = (0)$.*

Proof. When R is an integral domain, the statement of the theorem follows immediately from Proposition 1. On the other hand, suppose that R is a local ring with maximal ideal M, and set $J = \bigcap_{n\in\mathbb{N}} M^n$. Since R is Noetherian, J is a finitely generated R-module. As $MJ = J$, it follows from Nakayama's Lemma that $J = (0)$. Hence $\bigcap_{n\in\mathbb{N}} I^n \subseteq \bigcap_{n\in\mathbb{N}} M^n = (0)$. □

References

[1] H. Perdry: *An elementary proof of Krull's Intersection Theorem*, Amer. Math. Monthly **111** (2004) 356–357.

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