IDEAL THEORY AND PRÜFER DOMAINS

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Overrings of One-dimensional Noetherian Rings

The main purpose of this lecture is to prove that every overring of a one-dimensional Noetherian domain is again one-dimensional and Noetherian. Throughout this lecture, every ring is assumed to be commutative with an identity element.

Modules Over Noetherian Rings. In this subsection, we will established two results related to the set of annihilator of a finitely generated module over a Noetherian ring. For an R-module M, recall that $\text{Ann}(m) = \{r \in R : rm = 0\}$ is called the annihilator of m for every $m \in M$ and $\text{Ann}(M) = \{r \in R : rM = 0\}$ is called the annihilator of M. In this subsection we let $Z(M)$ denote the set $\bigcup_{m \in M \setminus \{0\}} \text{Ann}(m)$. We need the following lemma.

Lemma 1. *Let M be a nonzero R-module. Then the following statements hold.*

1. *If P is maximal in the set $\{\text{Ann}(m) : m \in M \setminus \{0\}\}$, then P is prime.*
2. *Every prime ideal minimal over $\text{Ann}(M)$ belongs to $Z(M)$.*

Proof. (1) Let $P = \text{Ann}(m)$ be maximal in the set $\mathscr{A} = \{\text{Ann}(m) : m \in M \setminus \{0\}\}$, and take $r, s \in R$ such that $rs \in P$. Then $r(sm) = 0$. If $sm = 0$, then $s \in P$. Otherwise, $\text{Ann}(sm)$ is an ideal in $\mathscr{A}$ containing r. Since $P \subseteq \text{Ann}(sm)$, the maximality of P ensures that $r \in \text{Ann}(sm) = P$. Thus, the ideal P is prime.

(2) Let P be a prime ideal minimal over $\text{Ann}(M)$. Consider the multiplicative subset $S := \{rt : r \notin Z(M) \text{ and } t \notin P\}$. Note that $R \setminus S \subseteq Z(M)$. Suppose, by way of contradiction, that $S \cap \text{Ann}(M)$ is nonempty and take $rt \in \text{Ann}(M) \cap S$ for some $r \in R \setminus Z(M)$ and $t \in R \setminus P$. Then $rtM = 0$. As $r \notin Z(M)$, the equality $tM = 0$ and so $t \in \text{Ann}(M) \subseteq P$, which is a contradiction. Thus, S is disjoint from $\text{Ann}(M)$. Let Q be maximal among all the ideals containing $\text{Ann}(M)$ and disjoint from S. Then $Q \subseteq Z(M)$ and $\text{Ann}(M) \subseteq Q \subseteq P$. It follows now from the minimality of P that $Q = P$ and, therefore, we can conclude that $P \subseteq Z(M)$. □

As we proceed to argue, in a zero-dimensional ring, every element that is not a zero-divisor must be a unit.

Proposition 2. *Let R be a zero-dimensional commutative ring with identity. If r is not a zero-divisor, then $r \in R^\times$.*

Proof. Suppose that r is not a zero-divisor. Considering R as a module over itself, we see that $r \notin Z(R)$ and $\text{Ann}(R) = 0$. Since R is zero-dimensional, every maximal ideal of R must be a minimal prime ideal over $\text{Ann}(R)$ and so must be included in $Z(R)$ by part (2) of Lemma 1. Hence r is not contained in any maximal ideal and, therefore, r must be a unit. □

Proposition 3. *Let R be a Noetherian ring and let M be a finitely generated R-module. Then there are only finitely many ideals of R that are maximal in $Z(M)$. Moreover, each of these ideals is a prime ideal of the form* $\text{Ann}(m)$ *for some nonzero $m \in M$.*

Proof. Since R is Noetherian and M is finitely generated, M is a Noetherian R-module. Let $\mathscr{M}$ be the set of maximal elements in $\big\{\text{Ann}(m) : m \in M \setminus \{0\}\big\}$. We know that every ideal in $\mathscr{M}$ is a prime ideal, and it is clear that the set $\mathscr{Z}$ of elements of R annihilating some nonzero element of M is the union of the prime ideals in $\mathscr{M}$. Write $\mathscr{M} = \{\text{Ann}(m) : m \in S\}$ for a subset S of M, and let us verify that $\mathscr{M}$ is finite. Let N denote the R-submodule of M spanned by S. As M is Noetherian, N is finitely generated. Write $N = Rm_1 + \cdots + Rm_n$ for some $m_1, \dots, m_n \in S$. Then for any $m \in S$ we can write $m = r_1 m_1 + \cdots + r_n m_n$, from which we obtain the inclusion $\text{Ann}(m_1) \cap \cdots \cap \text{Ann}(m_n) \subseteq \text{Ann}(m)$. Because $\text{Ann}(m)$ is prime, $\text{Ann}(m_i) \subseteq \text{Ann}(m)$ for some $i \in [\![1, n]\!]$. Now the maximality of $\text{Ann}(m_i)$ implies that $\text{Ann}(m) = \text{Ann}(m_i)$. As a consequence, $\mathscr{M}$ is finite. Finally, suppose that I is an ideal contained in $Z(M)$. Then $I \subseteq \bigcup_{i=1}^n \text{Ann}(m_i)$, and the fact that each $\text{Ann}(m_i)$ is prime implies that $I \subseteq \text{Ann}(m_j)$ for some $j \in [\![1, n]\!]$. □

Proposition 4. *Let R be a Noetherian ring, and let M be a finitely generated nonzero R-module. If P is a prime ideal of R minimal over* $\text{Ann}(M)$*, then $P =$* $\text{Ann}(m)$ *for some $m \in M$.*

Proof. Set $A = \text{Ann}(M)$. It is clear that the R_P-module M_P is a finitely generated module over the Noetherian ring R_P. Let us argue that M_P is nonzero. After writing $M = Rm_1 + \cdots + Rm_k$ for nonzero elements $m_1, \dots, m_k \in M$, we can see that $\bigcap_{i=1}^k \text{Ann}(m_k) \subseteq A \subseteq P$. This, together with the fact that P is prime, allows us to assume that $\text{Ann}(m) \subseteq P$ for a nonzero $m \in M$. Suppose towards a contradiction that $m/1 = 0/1$ in M_P. Then there must be an element $s \in R \setminus P$ with $sm = 0$. Therefore $s \in \text{Ann}(m) \subseteq P$, a contradiction. Thus, $m/1$ is nonzero in M_P, and so M_P is a nonzero R_P-module.

It is clear that A_P is contained in the annihilator of M_P; indeed, it equals the annihilator of M_P (see Exercise 1), but we do not use this fact in this proof. Let us verify that P_P is minimal over A_P. A prime ideal of R_P between A_P and P_P must have the form Q_P, where Q is a prime ideal of R such that $Q \subseteq P$. Observe that $A \subseteq {}^c(A_P) \subseteq {}^c(Q_P) = Q$ and $Q = {}^c(Q_P) = {}^c(P_P) = P$, where ${}^cJ$ denotes the contraction of an ideal J of R_P under the localization homomorphism $R \to R_P$. Since

$A \subseteq Q \subseteq P$, the minimality of P ensures that $Q = P$, that is, $Q_P = P_P$. Thus, P_P is minimal over A_P.

It follows now from part (2) of Lemma 1 that P_P is contained in $Z(M_P)$. Since P_P is the maximal ideal of R_P, we see that P_P is, in particular, maximal in the set $Z(M_P)$. As a consequence, Proposition 3 guarantees the existence of an element in M_P whose annihilator is P_P, and we can readily verify that such an element can be taken to be $m/1$ for some $m \in M$. Writing $P = Ra_1 + \cdots + Ra_n$ for some $a_1, \dots, a_n \in R$, we see that P_P is generated by the set $\{a_i/1 : i \in [\![1, n]\!]\}$. As $P_P = \text{Ann}(m/1)$, for every $i \in [\![1, n]\!]$ there is an $s_i \in R \setminus P$ such that $s_i a_i m = 0$. Then for $s = s_1 \cdots s_n$, the equality $sPm = 0$ holds. Finally, we claim that $P = \text{Ann}(sm)$. It is clear that P annihilates sm. Conversely, if $r \in R$ annihilates sm, then $rs/1$ annihilates $m/1$, and so $rs/1 \in P_P$, that is, $r \in P$. Hence $P = \text{Ann}(sm)$, which concludes the proof. □

Overrings of One-dimensional Noetherian Domains. In order to prove Theorem 9, we need to introduce the notion of length for modules. Let M be an R-module. A *composition series* of M is a chain

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_\ell = 0, \tag{0.1}$$

where M_j/M_{j+1} is simple, that is, M_j/M_{j+1} has no nonzero proper R-submodule for any $j \in [\![0, \ell - 1]\!]$. In this case, we say that the composition series (0.1) has *length* ℓ. The Jordan-Hölder Theorem states that if M has a composition series, then any chain of R-submodules can be refined to obtain a composition series of M, and that any two composition series of M have the same length. If M has a composition series like (0.1), then ℓ is called the *length* of M.

Recall that the Jacobson radical of R is the intersection of all maximal ideals of R. The following lemma will be used in the proof of Theorem 6.

Lemma 5. *Let R be a zero-dimensional Noetherian ring with identity. Then R has finitely many prime ideals, and* $\text{Rad}(0)$ *is the Jacobson radical of R.*

Proof. Since R is Noetherian, we know from previous lectures that $\text{Rad}\,(0)$ is the intersection of finitely many prime ideals, namely, $P_1, \dots, P_k$ (assume they are different). Since every prime ideal P of R contains $\text{Rad}\,(0)$, we see $P_1 \cdots P_k \subseteq P$. As P is prime there $P_j \subseteq P$ for some $j \in [\![1, k]\!]$, and the fact that R has dimension zero ensures that $P = P_j$. Therefore R has only finitely many prime ideals, which are also maximal ideals. Thus, $J := \text{Rad}\,(0)$ is the Jacobson radical of R. □

We are in a position to characterize zero-dimensional Noetherian rings in terms of composition series.

Theorem 6. *For a commutative ring R with identity, the following statements are equivalent.*

(a) *R is Noetherian and zero-dimensional.*

(b) *Every finitely generated R-module has a composition series.*

(c) *As an R-module, R has a composition series.*

Proof. (a) $\Rightarrow$ (b): Let M be a finitely generated R-module. The fact that R is Noetherian guarantees that J is finitely generated and so nilpotent. Thus, $(P_1 \cdots P_k)^m = 0$ for some $m \in \mathbb{N}$. Consider the ideals $I_1, \dots, I_{km}$ of R defined by $I_{qm+r} = (P_1 \dots P_q)^m P_{q+1}^r$, where $q \in [\![0, k-1]\!]$ and $r \in [\![1, m]\!]$. It is clear that $M \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_{km} = 0$, where $M_j := I_j M$. Now fix $j \in [\![1, km-1]\!]$. Since M_j/M_{j+1} is a finitely generated module over R/P for some $P \in \operatorname{Spec}(R)$ (i.e., M_j/M_{j+1} is a finite-dimensional vector space over the field R/P), there are (R/P)-submodules $M_{j,1}, \dots, M_{j,n_j}$ of M_j containing M_{j+1} such that $M_j = M_{j,1} \supsetneq M_{j,2} \supsetneq \cdots \supsetneq M_{j,n_j} = M_{j+1}$ satisfying that, for any $i \in [\![1, n_j - 1]\!]$, the quotient $M_{j,i}/M_{j,i+1}$ contains no nontrivial proper (R/P)-submodule and so no nontrivial proper R-submodule. Hence M has a composition series.

(b) $\Rightarrow$ (c): This is clear.

(c) $\Rightarrow$ (a): Since R has a composition series, it has finite length ℓ. Now if $(J_n)_{n \in \mathbb{N}}$ were an ascending chain condition of ideals (where $J_n \subsetneq J_{n+1}$), then Jordan-Hölder Theorem would allow us to refine the chain $R \supsetneq J_{\ell+1} \supsetneq J_\ell \supsetneq \cdots \supsetneq J_1 \supseteq (0)$ to obtain a composition series of R with length at least $\ell + 1$. Hence every ascending chain of ideals of R eventually stabilizes, and so R is Noetherian.

Let us finally argue that R has dimension 0. Let P be a prime ideal of R. Since R has a composition series, the integral domain R/P has a composition series as an R/P-module. Thus, it suffices to argue that every integral domain D with a composition series is a field. To prove this, let d be a nonzero element in D. Since D has a composition series, it must have a minimal nonzero ideal I. As $dI \subseteq I$, the minimality of I ensures that $dI = I$. Therefore $d \in I$, and so $d = da$ for some $a \in I$, which implies that $a = 1$. Hence$I = D$, and we can conclude that D is a field. □

Theorem 6 can be used to prove the following result.

Proposition 7. *Let R be an integral domain. Then R is Noetherian with dimension at most 1 if and only if the R-module R/I has a composition series for every nonzero ideal I.*

Proof. For the direct implication, assume that R is Noetherian with $\dim R \le 1$, and let I be a nonzero ideal of R. If $\dim R = 0$, then R is a field, and R/I is the zero R-module, which trivially has a composition series. Therefore we suppose that $\dim R = 1$. Observe that the ring R/I is zero-dimensional: indeed, if P is a minimal prime ideal over I, then the fact that $\dim R = 1$ ensures that P is maximal. Then R/I has a composition series as an (R/I)-module by Theorem 6, and so it has a composition series as an R-module.

For the reverse implication, assume that R/I has a composition series for every nonzero ideal I. If P is a nonzero prime ideal, then R/P has a composition series, and so Theorem 6 guarantees that the integral domain R/P is a zero-dimensional and so a field, whence P is maximal. Hence $\dim R = 1$. Finally, let $(I_n)_{n \in \mathbb{N}_0}$ be an ascending chain of ideals of R with $I_0 \neq (0)$. Then $(I_n/I_0)_{n\in\mathbb{N}}$ is an ascending chain of ideals of R/I_0. It follows now from Theorem 6 that R/I_0 is Noetherian, and so $(I_n/I_0)_{n\in\mathbb{N}}$ eventually stabilizes. Thus, the same holds for $(I_n)_{n\in\mathbb{N}}$. Hence R is Noetherian. □

In the proof of Theorem 9, we will use the following technical lemma.

Lemma 8. *Let R be a one-dimensional integral domain, and let a and b be nonzero elements of R. If $J = \{x \in R \mid xa^n \in Rb \text{ for some } n \in \mathbb{N}\}$, then $J + Ra = R$.*

Proof. Exercise. □

We are in a position to prove that every overring of a one-dimensional Noetherian domain is both one-dimensional and Noetherian.

Theorem 9. *Let R be a one-dimensional Noetherian domain. Then every overring of R that is not a field, is a one-dimensional Noetherian domain.*

Proof. Let T be an overring of R that is not a field. Take a nonzero $a \in R$ and, for every $n \in \mathbb{N}$, set $I_n := a^nT \cap R + Ra$. It is clear that $(I_n)_{n\in\mathbb{N}}$ is a descending chain of ideals of R containing Ra. Since R is a one-dimensional Noetherian ring, the R-module R/Ra has a composition series by virtue of Proposition 7. Therefore the descending sequence $(I_n/aR)_{n\in\mathbb{N}}$ of R-submodules of R/Ra must eventually stabilize. Take $N \in \mathbb{N}$ such that $I_n = I_N$ for every $n \geq N$. We will argue that $T \subseteq a^{-N}R + aT$. To do so, take $t := b/c \in T$ for some $b, c \in R$, and then set

$$J := \{x \in R \mid xa^n \in Rc \text{ for some } n \in \mathbb{N}\}.$$

In light of Lemma 8, the equality $R = J + Ra$ holds. So we can write $1 = j + ra$ for some $j \in J$ and $r \in R$. Take $k \in \mathbb{N}$ such that $ja^k \in Rc$. Now we see that $jt = a^{-k}b(ja^k/c) \in a^{-k}R$. Therefore $t = (j + ra)t = jt + rat \in a^{-k}R + aT$. Now take the minimum $m \in \mathbb{N}$ such that $t \in a^{-m}R + aT$.

We claim that $m \leq N$. Suppose, by way of contradiction, that $m > N$. Take $r_1 \in R$ and $t_1 \in T$ such that $t = a^{-m}r_1 + at_1$. Then $r_1 = a^m(t - at_1) \in a^mT$, and so $r_1 \in a^mT \cap R \subseteq I_m$. Since $m > N$, it follows that $I_m = I_{m+1}$, whence we can write $r_1 = a^{m+1}t_2 + r_2a$. Hence

$$t = \frac{r_1}{a^m} + at_1 = \frac{a^{m+1}t_2 + r_2a}{a^m} + at_1 = \frac{r_2}{a^{m-1}} + a(t_1 + t_2) \in a^{-(m-1)}R + aT.$$

However, this contradicts the minimality of m. As a consequence, $m \leq N$, as desired.

Because $t \in a^{-m}R + aT \subseteq a^{-N}R + aT$, the inclusion $T \subseteq a^{-N}R + aT$ holds. Therefore T/aT is a submodule of a cyclic R-module. As a result, T/aT is a finitely generated R-module. As any nonzero ideal of T contains a nonzero element of R, every quotient of T

by a nonzero ideal has a composition series. Hence T is a one-dimensional Noetherian domain by Proposition 7. $\square$

In general, an overring of a Noetherian domain does not have to be Noetherian, as the following example illustrates.

Example 10. Let R be the ring $\mathbb{Q}[x,y]$, which is a two-dimensional Noetherian domain with quotient field $\mathbb{Q}(x,y)$. Now consider the ring $T = \mathbb{Q}[x] + y\mathbb{Q}[x]_x[y]$, where $\mathbb{Q}[x]_x$ is the localization of $\mathbb{Q}[x]$ at the multiplicative set $\{x^n : n \in \mathbb{N}_0\}$ (i.e., the ring of Laurent polynomials $\mathbb{Q}[x,x^{-1}]$). It is clear that T is an overring of R. To argue that T is not Noetherian, it suffices to show that the ideal Ty is not finitely generated. Suppose, otherwise, that $Ty = (f_1, \dots, f_n)$. Take $m \in \mathbb{N}_0$ such that $x^m f_i \in \mathbb{Q}[x,y]$ for all $i \in [\![1,n]\!]$. Since $y/x^{m+1} \in Ty$, we can take $g_1, \dots, g_n \in T$ such that the equality

$$x^{-1}y = g_1 x^m f_1 + \cdots + g_n x^m f_n \tag{0.2}$$

holds. Then we can equate the coefficients of y in both sides of (0.2) to find that

$$x^{-1} = g(x,0)x^m \frac{d}{dy} f_1(x,0) + \cdots + g(x,0)x^m \frac{d}{dy} f_n(x,0) \in \mathbb{Q}[x],$$

where $\frac{d}{dy}h(x,y)$ denotes the formal derivative of $h \in \mathbb{Q}(x)[y]$ with respect to y. As this is clearly a contradiction, we conclude that T is not Noetherian.

Exercises

Exercise 1. *Let R be a commutative ring with identity, and let M be a finitely generated R-module. For a multiplicative subset S of R, prove that*

$$S^{-1}\mathrm{Ann}(M) = \mathrm{Ann}(S^{-1}M).$$

Exercise 2. *Let R be a one-dimensional integral domain, and let a and b be nonzero elements of R. Show that if $J = \{x \in R \mid xa^n \in Rb \text{ for some } n \in \mathbb{N}\}$, then $J + Ra = R$.*

Exercise 3. *Let R be a one-dimensional Noetherian domain, and let T be an overring of R. For a prime ideal P of R, show that there are only finitely many ideals Q of T lying over P, that is, satisfying $Q \cap R = P$.*

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