

IDEAL THEORY AND PRÜFER DOMAINS

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LECTURE 6: VALUATION DOMAINS I

For an integral domain R , we let $\text{qf}(R)$ denote the quotient field of R .

Definition 1. An integral domain R is called a *valuation domain* if for every nonzero $x \in \text{qf}(R)$, either x or x^{-1} belongs to R .

We observe that if R is a valuation domain, then for all nonzero $x, y \in R$ either $xy^{-1} \in R$ or $yx^{-1} \in R$, which means that $y \mid_R x$ or $x \mid_R y$.

It is clear from the definition that every field is a valuation domain. In addition, if R is a valuation domain and S is an extension ring of R with $R \subseteq S \subseteq \text{qf}(R)$, then S is also a valuation domain. Let us take a look at another example.

Example 2. For $p \in \mathbb{P}$, and consider the localization $\mathbb{Z}_{(p)}$ of \mathbb{Z} at the prime ideal (p) . The quotient field of $\mathbb{Z}_{(p)}$ is \mathbb{Q} . It is clear that for every nonzero rational number q , either $q \in \mathbb{Z}_{(p)}$ or $q^{-1} \in \mathbb{Z}_{(p)}$. As a result, $\mathbb{Z}_{(p)}$ is a valuation domain.

Example 3. What are the valuation domains of $\mathbb{C}(x)$? As in the previous example, we can readily see that $\mathbb{C}[x]_{(x-\alpha)}$ for any $\alpha \in \mathbb{C}$ are valuation domains of $\mathbb{C}(x)$. Indeed, these are the only valuation rings of $\mathbb{C}(x)$ containing $\mathbb{C}[x]$ (prove this!). However, there is a valuation domain of $\mathbb{C}(x)$ that does not contain $\mathbb{C}[x]$: the integral domain $R := \mathbb{C}[x^{-1}]_{(x^{-1})}$. To show that R is a valuation domain it suffices to observe that $x \mapsto x^{-1}$ induces an automorphism of $\mathbb{C}(x)$ sending $\mathbb{C}[x]_{(x)}$ to R .

A *fractional ideal* J of an integral domain R is an R -submodule of $\text{qf}(R)$ for which there exists a nonzero $r \in R$ such that rJ is an ideal of R . A fractional ideal J is called *principal* if there exists $x \in \text{qf}(R)$ such that $J = Rx$. In particular, every ideal (resp., principal ideal) of an integral domain is a fractional ideal (resp., principal fractional ideal).

Example 4. If R is an integral domain, and J is a finitely generated R -submodule of $\text{qf}(R)$, then J is a fractional ideal. To check this, suppose that $J = Rq_1 + \cdots + Rq_n$ for some $q_1, \dots, q_n \in J$. For each $i \in \llbracket 1, n \rrbracket$, we can write $q_i = a_i/b_i$ for some $a_i, b_i \in R$ with $b_i \neq 0$. After setting $b = b_1 \cdots b_n$, we see that $bq_1, \dots, bq_n \in R$. As a result, $bJ = Rbq_1 + \cdots + Rbq_n$ is an ideal of R . Hence J is a fractional ideal.

We can characterize a valuation domain in terms of its poset of (principal) ideals.

Proposition 5. *For an integral domain R , the following statements are equivalent.*

- (a) R is a valuation domain.
- (b) The principal ideals of R are totally ordered by inclusion.
- (c) The ideals of R are totally ordered by inclusion.
- (d) The principal fractional ideals of R are totally ordered by inclusion.
- (e) The fractional ideals of R are totally ordered by inclusion.

Proof. (a) \Leftrightarrow (b): For all $x, y \in R$, it is clear that $x \mid_R y$ if and only if $yR \subseteq xR$, whence the equivalence follows.

(b) \Leftrightarrow (c): Clearly, (c) implies (b). For the direct implication, suppose that I_1 and I_2 are ideals of R such that $I_1 \not\subseteq I_2$. Take $x \in I_1 \setminus I_2$. Now take $y \in I_2$, and note that $y \nmid_R x$, which means that $xR \not\subseteq yR$. Since the principal ideals of R are totally ordered, $y \in yR \subseteq xR \subseteq I_1$. Thus, $I_2 \subseteq I_1$.

(c) \Rightarrow [(d) and (e)]: It suffices to check that (c) implies (e). Let J_1 and J_2 be two fractional ideals of R . Take $r_1, r_2 \in R$ such that r_1J_1 and r_2J_2 are ideals of R . Then $r_1r_2J_1$ and $r_1r_2J_2$ are also ideals of R , and so they must be comparable. Hence the fractional ideals J_1 and J_2 are also comparable.

[(d) or (e)] \Rightarrow (b): This is obvious. □

Corollary 6. *Every valuation domain is a local ring.*

Proof. It follows from Proposition 5 that the ideals of R are totally ordered by inclusion. Therefore R must contain exactly one maximal ideal, namely, the union of all proper ideals. □

An integral domain R is called a *Bezout domain* if every finitely generated ideal of R is principal. Valuation domains can be characterized as the local Bezout domains.

Let us establish a further characterization of a valuation domain.

Proposition 7. *An integral domain is a valuation domain if and only if it is a local Bezout domain.*

Proof. For the direct implication, suppose that R is a valuation domain. Then R is local by Corollary 6. To show that R is a Bezout domain, let $I = (r_1, \dots, r_n)$ be a finitely generated ideal of R . Since $\{r_jR : j \in \llbracket 1, n \rrbracket\}$ is a totally ordered, it has a maximum element, namely, rR for some $r \in \{r_1, \dots, r_n\}$. It is clear that $I = rR$. Hence R is a local Bezout domain.

For the reverse implication, suppose that R is a local Bezout domain. Take $a, b \in R$, and consider the ideal $I := Ra + Rb$. Let M be the maximal ideal of R . One can easily verify that the abelian group I/MI is indeed an R/M -module, that is, a vector space over the field R/M . Since I is finitely generated, it is principal and, therefore, the vector space I/MI has dimension one. So there exist $u, v \in R$ such that $ua + vb \in MI$, where either $u \in R^\times$ or $v \in R^\times$. So we can take $r, s \in M$ such that $ua + vb = ra + sb$,

that is, $(u - r)a = (y - s)b$. Since $u - r \in R^\times$ or $y - s \in R^\times$, it follows that either $b \mid_R a$ or $a \mid_R b$. Hence R is a valuation domain. \square

Corollary 8. *Every Noetherian valuation domain is a PID.*

A pair (G, \leq) , where G is an additive abelian group and \leq is an order relation on G , is called an *ordered group* provided that \leq is translation-invariant, that is, for all $a, b, c \in G$, the inequality $b \leq c$ implies that $a + b \leq a + c$. If \leq is a total order, then we say that (G, \leq) is a *totally ordered group* or a *linearly ordered group*. To easy notation we often write G instead of (G, \leq) . For an ordered group G , the set $G_+ := \{a \in G : a \geq 0\}$ is a submonoid of G , which is called the *nonnegative cone* of G . If G is a totally ordered group, then it follows immediately that, for every $a \in G \setminus \{0\}$, exactly one of the inclusions $a \in G_+$ and $-a \in G_+$ holds.

For a field F and a totally ordered (abelian) group G , a map $v: F \rightarrow G \cup \{\infty\}$, where ∞ is a symbol not in G such that $x \leq \infty$ and $x + \infty = \infty$ for all $x \in G$, is called a *valuation map* if the following conditions hold:

- (1) $v(0) = \infty$,
- (2) $v: F^\times \rightarrow G$ is a group homomorphism, and
- (3) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in F$.

Here is the characterization of a valuation domain that suggests the chosen terminology.

Theorem 9. *For an integral domain R , the following statements are equivalent.*

- (a) R is a valuation domain.
- (b) There exists a valuation map $v: \text{qf}(R) \rightarrow G \cup \{\infty\}$ such that

$$R = \{x \in \text{qf}(R) : v(x) \geq 0\}.$$

Also, if v and R are as in part (b), then the maximal of R is $\{x \in \text{qf}(R) : v(x) > 0\}$.

Proof. (a) \Rightarrow (b): Assume that R is a valuation domain, and set $G := \text{qf}(R)^\times / R^\times$. We will write G additively, that is, $xR^\times + yR^\times = xyR^\times$ for all $x, y \in \text{qf}(R)^\times$. Now define the binary relation \leq on G as follows: $xR^\times \leq yR^\times$ if $yx^{-1} \in R$. Using that $yx^{-1} \in R$ if and only if $Ry \subseteq Rx$ for all $x, y \in \text{qf}(R)^\times$, one can readily check that \leq is an order relation on G compatible with the addition, and it follows from Proposition 5 that G is indeed a totally ordered group. Also, we see that $G_+ = \{xR^\times : x \in R \setminus \{0\}\}$.

Now we can define $v: \text{qf}(R) \rightarrow G \cup \{\infty\}$ by $v(0) = \infty$ and $v(x) = xR^\times$ if $x \neq 0$. It is clear that $v: \text{qf}(R)^\times \rightarrow G$ is a group homomorphism. Observe that the inequality $v(x + y) \geq \min\{v(x), v(y)\}$ follows immediately when at least one of the elements x, y , and $x + y$ equals zero. Now take $x, y \in \text{qf}(R)^\times$ with $x + y \neq 0$ and assume, without loss

of generality, that $xR^\times \leq yR^\times$. Since $1 + y/x \in R$, the element $(1 + y/x)R^\times$ belongs to G_+ and, therefore,

$$v(x + y) = x(1 + y/x)R^\times = xR^\times + (1 + y/x)R^\times \geq xR^\times = v(x).$$

Hence $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in \text{qf}(R)$, which is the third defining condition of a valuation map. Thus, v is a valuation map. The last part of statement (b) can be readily deduced from the fact that $G_+ = \{xR^\times : x \in R \setminus \{0\}\}$.

(b) \Rightarrow (a): Suppose that $v: \text{qf}(R) \rightarrow G \cup \{\infty\}$ is a valuation map satisfying that $R = \{x \in \text{qf}(R) : v(x) \geq 0\}$. Take $x, y \in R$ to be nonzero elements. Then either $v(xy^{-1}) \in G_+$ or $v(yx^{-1}) \in G_+$, which implies that either $xy^{-1} \in R$ or $yx^{-1} \in R$. Hence $y \mid_R x$ or $x \mid_R y$. As a consequence, R is a valuation domain.

For the final statement, one can easily check that $M := \{x \in R : v(x) > 0\}$ is an ideal of R . If a nonzero $u \in R$ has a positive valuation, then $v(u^{-1}) = -v(u) < 0$, and so $u \notin R^\times$. In addition, if $v(u) = 0$, then $v(u^{-1}) = 0$, which means that $u \in R^\times$. Hence $R^\times = R \setminus M$, which means that M is a maximal ideal. \square

With notation as in the proof of Theorem 9, the group G is called the *group of divisibility* of the integral domain R .

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