IDEAL THEORY AND PRÜFER DOMAINS

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Integral Extensions II (Spectral Theorems)

In this lecture, we discuss three important results about lifting prime ideals in integral extensions. We assume that all rings we deal with here are commutative rings with identities.

Theorem 1 (Lying Over Theorem)**.** *Let $R \subseteq S$ be an integral ring extension. Then every prime ideal of R has the form $Q \cap R$ for some prime ideal Q of S.*

Proof. Let P be a prime ideal of R. Set $M = R \setminus P$. Since M is a submonoid of the multiplicative monoid of S, there exists an ideal Q of S that is maximal in the set of all ideals of S disjoint from M. In addition, we have seen that such an ideal Q is prime. It immediately follows that $Q \cap R \subseteq P$. Now assume, by way of contradiction, that there is an $r \in P$ such that $r \notin Q$. Since Q is properly contained in the ideal $Q + (r)$ of S, the maximality of Q ensures the existence of $m \in M$ such that $m \in Q + (r)$. Write $m = q + sr$ for some $q \in Q$ and $s \in S$. Since s is integral over R, there is an $n \in \mathbb{N}$ such that $s^n + \sum_{i=0}^{n-1} c_i s^i = 0$ for some $c_0, \dots, c_{n-1} \in R$. After multiplying this equality by r^n, we see that

$$(sr)^n + \sum_{i=0}^{n-1} c_i r^{n-i} (sr)^i = 0.$$

Substituting $sr = m - q$ in the previous equality and applying the Binomial Theorem, we obtain that $t := m^n + \sum_{i=0}^{n-1} c_i r^{n-i} m^i \in Q$. As a result, $t \in R \cap Q \subseteq P$. Since both t and r belong to P, then it follows that $m^n \in P$. Therefore $m \in P$, contradicting that P is disjoint from M. Hence $P \subseteq Q \cap R$, which completes the proof. □

With notation as in Theorem 1, we say that the ideal Q *lies over* P.

Theorem 2 (Going Up Theorem)**.** *Let $R \subseteq S$ be an integral ring extension, and let P_1 and P_2 be two prime ideals of R such that $P_1 \subseteq P_2$. If Q_1 is a prime ideal of S lying over P_1, then there exists a prime ideal Q_2 of S lying over P_2 such that $Q_1 \subseteq Q_2$.*

Proof. Since P_2 is a prime ideal of R, the set $M = R \setminus P_2$ is a submonoid of $S \setminus \{0\}$. As $P_1 = Q_1 \cap R$, the ideal Q_1 is disjoint from M. As a result, there exists a prime ideal Q_2 of S that is maximal among all ideal of S containing Q_1 and disjoint from M. We can now show that Q_2 lies over P_2 by mimicking the proof of Theorem 1. □

In an integral extension, not two prime ideals lying over the same prime ideal are comparable. Let us prove this assertion.

Theorem 3 (Incomparability Theorem)**.** *Let $R \subseteq S$ be an integral ring extension, and let Q_1 and Q_2 be two prime ideals of S such that $Q_1 \subseteq Q_2$. If $Q_1 \cap R = Q_2 \cap R$, then $Q_1 = Q_2$.*

Proof. Set $P = Q_1 \cap R$, which is a prime ideal of R. Let M be the submonoid $R \setminus P$ of $R \setminus \{0\}$. Now consider the collection $\mathscr{S}$ of all ideals I of S disjoint from M. We will argue that Q_1 is a maximal ideal in $\mathscr{S}$. Suppose, by way of contradiction, that this is not the case, and take an ideal I in $\mathscr{S}$ such that $Q_1 \subsetneq I$. Take $s \in I \setminus Q_1$. Since the extension $R \subseteq S$ is integral, there is a polynomial $f(x) = x^n + \sum_{i=0}^{n-1} c_i x^i$ in $R[x]$ of minimum degree such that $f(s) \in Q_1$. Because $c_0 = -\big(s^n + \sum_{i=1}^{n-1} c_i s^i\big) \in I$, it follows that $c_0 \in I \cap R \subseteq P = Q_1 \cap R \subseteq Q_1$. Therefore we see that $s\big(s^{n-1} + \sum_{i=1}^{n-1} c_i s^{i-1}\big) \in Q_1$. However, $s \notin Q_1$ and the minimality of $f(x)$ guarantees that $s^{n-1} + \sum_{i=1}^{n-1} c_i s^{i-1} \notin Q_1$, contradicting the fact that Q_1 is a prime ideal. Hence Q_1 is a maximal ideal in $\mathscr{S}$. As a result, if Q_2 is an ideal of S satisfying that $Q_1 \subseteq Q_2$ and $Q_1 \cap R = Q_2 \cap R$, then the equality $Q_1 = Q_2$ must hold. □

A chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of a ring R is said to have *length* n. In addition, the (*Krull*) *dimension* of R is the supremum of the lengths of all its chains of prime ideals. Clearly, every field has dimension 0, and a PID that is not a field has dimension 1. In addition, a polynomial ring $K[x_1, \dots, x_n]$ over a field K has dimension n (see [2, page 285]).

Corollary 4. *If $R \subseteq S$ is an integral extension, then R and S have the same dimension.*

Proof. It follows as an immediate consequence of the Lying Over, Going Up, and Incomparability Theorems. □

Integral extensions respect maximal ideals.

Proposition 5. *Let $R \subseteq S$ be an integral ring extension, and let Q be a prime ideal of S lying over a prime ideal P of R. The ideal Q is maximal if and only if the ideal P is maximal.*

Proof. Suppose first that Q is maximal, and take a maximal ideal P_1 of R with $P \subseteq P_1$. By the Going-Up Theorem, there is a prime ideal Q_1 of S containing Q and lying over P_1. Since Q is maximal, $Q_1 = Q$ and so $P_1 = Q_1 \cap R = Q \cap R = P$. Thus, P is maximal.

Conversely, suppose that P is maximal. Let Q_1 be a maximal ideal of S containing Q. Clearly, Q_1 lies over a prime ideal of R containing P, and so the maximality of P ensures that $Q_1 \cap R = P$. Since both Q and Q_1 lie over P, it follows from the Incomparability Theorem that $Q_1 = Q$. Hence Q is maximal. □

We conclude this lecture with a statement of a dual version of the Going Up Theorem.

Theorem 6 (Going Down Theorem)**.** *Let R be an integrally closed domain, and let S be an integral extension of R. If P_1 and P_2 are prime ideals of R such that $P_1 \subseteq P_2$ and Q_2 is a prime ideal of S lying over P_2, then there exists a prime ideal Q_1 of S which is contained in Q_2 and lies over P_1.*

Proof. See [1, Section 15.3]. □

Exercises

Exercise 1. *Let $R \subseteq S$ be an integral extension, and let Q be a maximal ideal of S lying over (the maximal ideal) P. Argue with a counterexample that S_Q may not be integral over R_P.*

Exercise 2. *Let F be a field with a subring R, and let P be a prime ideal of R. Prove that for any nonzero $a \in F$, either $R[a]$ or $R[1/a]$ contains a prime ideal lying over P.*

Exercise 3. *Let R be an integral domain with quotient field K. Let L be an algebraic extension of K, let T be the integral closure of R in L, and set $T_0 := T \cap K$. Prove that the following statements hold.*

(1) *T_0 is the integral closure of R.*
(2) *L is the quotient field of T.*
(3) *If $a \in T$ and $m(x) \in K[x]$ is the minimal polynomial of a, then $m(x) \in T_0[x]$.*

References

[1] D. S. Dummit and R. M. Foote: *Abstract Algebra* (Third Edition), John Wiley & Sons, 2004.
[2] D. Eisenbud: *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics vol. 150, Springer Verlag, New York 1999.

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