IDEAL THEORY AND PRÜFER DOMAINS

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Lecture 8: Prüfer Domains I

Throughout this section, R is an integral domain. Recall that $\mathrm{qf}(R)$ denotes the quotient field of R.

Definition and Examples. Prüfer domains, which are natural generalizations of valuation domains, play a fundamental role in multiplicative ideal theory. In this lecture, we start our discussion of Prüfer domains.

A fractional ideal I of R is *invertible* if there is a factional ideal J such that $IJ = R$, in which case $J = (R : I) = \{r \in \mathrm{qf}(R) : rI \subseteq R\}$. It is clear that the set of all invertible fractional ideals of R is an abelian group with identity R. Observe that such a group contains the set of all nonzero principal fractional ideals as a subgroup.

Definition 1. An integral domain R is a *Prüfer domain* if every nonzero finitely generated ideal of R is invertible.

Fields and PIDs are clearly Prüfer domains. Recall that a Bezout domain is an integral domain where every finitely generated ideal is principal. Since nonzero principal ideals are invertible, every Bezout domain is a Prüfer domain. In particular, every valuation domain is a Prüfer domain. Let us briefly exhibit two further examples of Prüfer domains.

Example 2. The set $\mathrm{Int}(R) := \{p(x) \in \mathbb{Q}[x] : p(\mathbb{Z}) \subseteq \mathbb{Z}\}$ is a subring of $\mathbb{Q}[x]$ called the *ring of integer-valued polynomials*. We shall prove soon enough that $\mathrm{Int}(R)$ is a non-Noetherian Prüfer domain of Krull dimension 2.

Example 3. The ring consisting of all the entire function on the complex plane is a Bezout domain of infinite Krull dimension. In particular, it is a Prüfer domain.

Although every PID is Prüfer, this is not the case for UFDs. The following example sheds some light upon this observation.

Example 4. For a field F, consider the ring of polynomials $R := F[x, y]$ and the ideal $I = Rx + Ry$ of R. If $f \in \mathrm{qf}(R)$ belongs to $J := (R : I)$, then $Rxf + Ryf \subseteq R$, and so $xf \in R$ and $yf \in R$. Therefore $f \in x^{-1}R \cap y^{-1}R = R$. Then $J \subseteq R$ (indeed, $J = R$), and we see that $IJ \subseteq I$. Thus, I is not an invertible ideal even though it is finitely generated, and this allows us to conclude that R is not a Prüfer domain. Note, however, that R is a UFD.

Characterizations. We will discuss various of the many characterizations of Prüfer domains. Let us start by the following.

Proposition 5. *For an integral domain R, the following statements are equivalent.*

(a) *R is a Prüfer domain.*

(b) *Every two-generated ideal of R is invertible.*

Proof. (a) $\Rightarrow$ (b): This is obvious.

(b) $\Rightarrow$ (a): We will show that every nonzero finitely generated ideal of R is invertible by using induction on the minimum number n of generators of such an ideal. It is clear when $n = 1$, and it follows from part (b) when $n = 2$. Suppose, therefore, that I can be generated by n elements, where $n > 2$, and assume that every nonzero ideal of R that can be generated by less than n elements is invertible. Now write $I = Rc_1 + \cdots + Rc_n$ for some nonzero elements $c_1, \dots, c_n \in R$. Set $I_1 := Rc_1 + \cdots + Rc_{n-1}$, $I_2 := Rc_2 + \cdots + Rc_n$, and $I_3 := Rc_1 + Rc_n$. By induction, I_1, I_2, and I_3 are invertible. Then $J := c_1 I_1^{-1} I_3^{-1} + c_n I_2^{-1} I_3^{-1}$ is a fractional ideal of R. We claim that J is the inverse of I. To show this, first observe that

$$\begin{aligned} IJ &= (I_1 + Rc_n)c_1 I_1^{-1} I_3^{-1} + (Rc_1 + I_2)c_n I_2^{-1} I_3^{-1} \\ &= c_1 I_3^{-1} + c_1 c_n I_1^{-1} I_3^{-1} + c_1 c_n I_2^{-1} I_3^{-1} + c_n I_3^{-1} \\ &= c_1 I_3^{-1}(R + c_n I_2^{-1}) + c_n I_3^{-1}(R + c_1 I_1^{-1}). \end{aligned}$$

As I_1 and I_2 are invertible ideals and $c_1 \in I_1$ and $c_n \in I_2$, it follows that $c_1 I_1^{-1} \subseteq R$ and $c_n I_2^{-1} \subseteq R$. This, along with the previous chain of equalities, guarantees that $IJ = c_1 I_3^{-1} + c_n I_3^{-1} = I_3 I_3^{-1} = R$. Hence I is an invertible ideal. $\square$

Before giving further characterizations of a Prüfer domain, we need to prove the following lemma.

Lemma 6. *For an integral domain R, the following statements hold.*

(1) *Every invertible (fractional) ideal of R is finitely generated.*

(2) *If R is local, then every invertible (fractional) ideal is principal.*

Proof. (1) Let I be an invertible (fractional) ideal of R. Take J to be the fractional ideal satisfying $IJ = R$, and write $1 = \sum_{i=1}^n a_i b_i$ for $a_1, \dots, a_n \in I$ and $b_1, \dots, b_n \in J$. Then for every $x \in I$, we see that $x = \sum_{i=1}^n a_i (x b_i)$. Since $x b_i \in R$ for every $i \in [\![1, n]\!]$, it follows that $x \in Ra_1 + \cdots + Ra_n$. So $I \subseteq Ra_1 + \cdots + Ra_n$. Since the reverse inclusion also holds, I is a finitely generated ideal.

(2) Let R be a local ring with maximal ideal M. Let I be an invertible (fractional) ideal of R with inverse J. As in the previous part, we can write $1 = \sum_{i=1}^n a_i b_i$ for $a_1, \dots, a_n \in I$ and $b_1, \dots, b_n \in J$. As $1 \notin M$, we see that $a_j b_j \notin M$ for some $j \in [\![1, n]\!]$. Since R is local, $a_j b_j \in R^\times$. Then for every $x \in I$, we obtain that $x = u(x b_j) a_j \in R a_j$,

where $u := (a_j b_j)^{-1} \in R$. Hence $I \subseteq Ra_j$. Since the reverse inclusion clearly holds, I is a principal ideal. ☐

We can now characterize Prüfer domains in terms of valuations.

Proposition 7. *For an integral domain R, the following statements are equivalent.*

(a) *R is a Prüfer domain.*

(b) *R_P is a valuation domain for every prime ideal P.*

(c) *R_M is a valuation domain for every maximal ideal M.*

Proof. (a) $\Rightarrow$ (b): Assume that R is a Prüfer domain. Since R_P is a local ring, it is enough to prove that it is a Bezout domain. Let I_P be a nonzero finitely generated ideal of R_P. Then the contraction I of I_P is a nonzero finitely generated ideal of R and, therefore, it is invertible. If J is a fractional ideal such that $JI = R$, then $(JR_P)I_P = R_P$, and so I_P is invertible in R_P. Thus, it follows from Lemma 6 that I_P is a principal ideal.

(b) $\Rightarrow$ (c): This is clear.

(c) $\Rightarrow$ (a): Let I be a nonzero proper finitely generated ideal of R. Then I is contained in a maximal ideal M. Since I is finitely generated, the ideal extension I_M of I in R_M is also finitely generated. This, along with the fact that R_M is a Bezout domain (because it is Prüfer), implies that I_M is a nonzero principal ideal. Hence I must be invertible. As a result, R is a Prüfer domain. ☐

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