IDEAL THEORY AND PRÜFER DOMAINS

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Lecture 9: Prüfer Domains II

Throughout this section, R is an integral domain. Recall that $\text{qf}(R)$ denotes the quotient field of R.

Valuation Characterizations of Prüfer Domains. We begin this lecture by characterizing Prüfer domains in terms of valuation domains.

Proposition 1. *For an integral domain R, the following statements are equivalent.*

(a) *R is a Prüfer domain.*

(b) *R_P is a valuation domain for every prime ideal P.*

(c) *R_M is a valuation domain for every maximal ideal M.*

Proof. (a) $\Rightarrow$ (b): Assume that R is a Prüfer domain, and let P be a prime ideal of R. Since R_P is a local ring, it is enough to prove that it is a Bezout domain. Let I_P be a nonzero finitely generated ideal of R_P. Then the contraction I of I_P is a nonzero finitely generated ideal of R and, therefore, it is invertible. If J is a fractional ideal such that $JI = R$, then $(JR_P)I_P = R_P$, and so I_P is invertible in R_P. We have seen in a previous lecture that, in a local domain, invertible ideals must be principal. Thus, I_P is a principal ideal. Hence R_P is a valuation domain.

(b) $\Rightarrow$ (c): This is clear.

(c) $\Rightarrow$ (a): Assume, by way of contradiction, that there is a nonzero finitely generated ideal I of R that is not invertible. Write $I = Ra_1 + \cdots + Ra_n$ for $a_1, \dots, a_n \in R$. Since I is not invertible, $IJ \subsetneq R$, where $J := (R : I)$. So there is a maximal ideal M of R such that $IJ \subseteq M$. Because the extension I_M of I is a finitely generated ideal of the Bezout domain R_M, there is an $a \in I$ satisfying $I_M = aR_M$. For each $i \in [\![1, n]\!]$, we can now take $s_i \in R \setminus M$ with $s_i a_i \in aR$. After setting $s = s_1 \cdots s_n$, we see that $sa^{-1}a_i \in R$ for every $i \in [\![1, n]\!]$, and so $sa^{-1}I \subseteq R$. This implies that $sa^{-1} \in J$ and, therefore, $s = (sa^{-1})a \in JI \subseteq M$, which is a contradiction. $\square$

Dedekind Domains. It is natural to wonder which Prüfer domains are Noetherian domains. Noetherian Prüfer domains are perhaps the best studied and understood class of Prüfer domains; they are called Dedekind domains. We will use, however, a more standard definition.

Definition 2. An integral domain is a *Dedekind domain* if it is an integrally closed Noetherian domain of dimension at most 1.

Fields are clearly the zero-dimensional Dedekind domains. On the other hand, PIDs are Dedekind domains. Rings of algebraic integers are also relevant examples of Dedekind domains. We proceed to establish some useful characterizations of Dedekind domains.

Theorem 3. *For an integral domain R that is not a field, the following statements are equivalent.*

(a) *R is a Dedekind domain.*
(b) *R is Noetherian and R_P is a Noetherian valuation domain for every prime ideal P of R.*
(c) *Every nonzero ideal of R is invertible.*

Proof. (a) $\Rightarrow$ (b): As Dedekind domains are Noetherian by definition, it suffices to prove that R_P is a PID for every prime ideal P. Let P be a prime ideal of R. After replacing R by R_P, we can assume that R is local Dedekind domain with maximal ideal P. Let Rx be maximal among all the principal ideals contained in P. To show that $P = Rx$, we only need to argue that $P \subseteq Rx$. Suppose, by way of contradiction, that this is not the case. Since R is a 1-dimensional local domain, $\operatorname{Rad} Rx = P$, and so the fact that P is finitely generated guarantees the existence of a minimum $m \in \mathbb{N}$ such that $P^m \subseteq Rx$. Take $y \in P^{m-1}$ such that $y \notin Rx$ (note that $m \ge 2$). Then $y/x \in \operatorname{qf}(R)$ satisfies that $y/x \notin R$ but $(y/x)P \subseteq R$. Since $(y/x)P$ is an ideal of R, either $(y/x)P = R$ or $(y/x)P \subseteq P$.

CASE 1: $(y/x)P = R$. In this case, we can take $r \in P$ such that $yr = x$. Since $r \notin R^\times$, it follows that $Rx \subsetneq Ry$, which contradicts the maximality of Rx.

CASE 2: $(y/x)P \subseteq P$. Set $s = y/x$. Since R is Noetherian, we can take nonzero elements $a_1, \dots, a_n \in R$ such that $P = Ra_1 + \dots + Ra_n$. As $sP \subseteq P$, for every $j \in [\![1, n]\!]$ we can write $sa_j = \sum_{i=1}^n c_{ij} a_i$ for some $c_{ij}, \dots, c_{ij} \in R$. Equivalently, $Mv = 0$, where M is the matrix $(\delta_{ij} s - c_i j)_{1 \le i,j \le n}$ and v is the vector $(a_1, \dots, a_n)^T$. By Cramer's Rule, $(\det M)a_1 = 0$. So $\det M = 0$, which implies that $s = y/x$ is a root of the monic polynomial $\det(tI_n - C) \in R[t]$, where $C = (c_{ij})_{1 \le i,j \le n}$. Hence y/x is integral over R. Since R is integrally closed, $y/x \in R$, which is a contradiction.

Thus, $P = Rx$. Let I be a proper ideal of R. By Krull's Intersection Theorem, $\bigcap_{n \in \mathbb{N}} P^n = (0)$, and so there an $n \in \mathbb{N}$ such that $I \subseteq P^n$ but $I \nsubseteq P^{n+1}$. Take $a \in I \setminus P^{n+1}$, and write $a = ux^n$ for some $u \in R$. Since $a \notin P^{n+1}$, we obtain that

$u \notin P$. This implies that $u \in R^\times$, and so $x^n = u^{-1}a \in I$, that is, $I = P^n$. Hence R is a PID, as desired.

(b) $\Rightarrow$ (c): Suppose, by way of contradiction, that there is a nonzero ideal I of R that is not invertible. Since R is Noetherian, $I = Ra_1 + \cdots + Ra_n$ for some $a_1, \dots, a_n \in R$. Because I is not invertible, $IJ \subsetneq R$, where $J = \{r \in \text{qf}(R) : rI \subseteq R\}$. Let P be a maximal ideal of R such that $IJ \subseteq P$. By (b), the ideal extension I_P of I is principal in R_P. Now we can mimic the proof of (c) $\Rightarrow$ (a) of Proposition 1 to obtain a contradiction.

(c) $\Rightarrow$ (a): We have seen before that every invertible ideal is finitely generated. Thus, R is Noetherian. Now suppose that M is a maximal ideal of R. We can easily verify that the extension of any invertible ideal of R is invertible in R_M. Then every nonzero ideal of R_M is invertible. Since every invertible ideal of a local ring is principal (see previous lectures), R_M is a PID. Hence R has dimension 1. Finally, as every PID is integrally closed, R_M is integrally closed for every maximal ideal M. As a result, R must be integrally closed. Thus, we can conclude that R is a Dedekind domain. □

The Noetherian Prüfer domains are precisely the Dedekind domains, as the following corollary indicates.

Corollary 4. *An integral domain is a Dedekind domain if and only if it is a Noetherian Prüfer domain.*

Proof. The corollary clearly holds for fields. Let R be an integral domain that is not a field. If R is a Dedekind domain, then R is Noetherian, and Theorem 3 ensures that R_P is a PID for every prime ideal P. Since every PID is a valuation domain, R is a Prüfer domain, and the direct implication follows. For the reverse implication, it suffices to observe that in a Noetherian Prüfer domain every nonzero ideal is finitely generated and so invertible, whence we are done by virtue of Theorem 3. □

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