

PROBLEM SET 4: PRECISE CONCEPTS OF LIMIT AND CONTINUITY

Note: Most of the problems were taken from the textbook [1].

Problem 1. Prove the following statements using the $\epsilon - \delta$ definition of limit.

a) $\lim_{x \rightarrow 1} \frac{2+4x}{3} = 2;$

b) $\lim_{x \rightarrow a} x = a;$

c) $\lim_{x \rightarrow 0} x^3 = 0;$

d) $\lim_{x \rightarrow 2} (x^2 - 4x + 5) = 1;$

e) $\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}.$

Problem 2. If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases},$$

then prove that $\lim_{x \rightarrow 0} f(x)$ does not exist.

Problem 3. Suppose that $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = c$, where $c > 0$. Prove that

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \infty \quad \text{and} \quad \lim_{x \rightarrow a} [f(x)g(x)] = \infty.$$

Problem 4. Why are the following functions continuous at every point of its domain. State the domain.

a) $F(x) = \sin(\cos(\sin x));$

b) $B(x) = \frac{\tan x}{\sqrt{4-x^2}};$

c) $N(x) = \tan^{-1}(1 + e^{-x^2}).$

Problem 5. Use continuity to evaluate $\lim_{x \rightarrow 2} x\sqrt{20-x^2}$ and $\lim_{x \rightarrow 4} 3\sqrt{x^2-2x-4}$.

Problem 6. Sketch the graph of the following function:

$$f(x) = \begin{cases} 1-x^2 & \text{if } x < 1 \\ 1/x & \text{if } x \geq 1 \end{cases}.$$

Is the function f continuous everywhere? Explain.

Problem 7. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

Problem 8. Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

a) $x^2 + x - 3 = 0$, $(1, 2)$;

b) $\sqrt[3]{x} = 1 - x$, $(0, 1)$;

c) $\sin x = x^2 - x$, $(1, 2)$.

Problem 9. Prove that each of the following equations has at least one real root.

a) $\cos x = x^3$;

b) $x^5 - x^2 + 2x + 3 = 0$;

c) $x^5 - x^2 - 4 = 0$.

Problem 10. Find the limit or show it does not exist.

a) $\lim_{x \rightarrow \infty} \frac{(2x^2+1)^2}{(x-1)^2(x^2+x)}$;

b) $\lim_{x \rightarrow \infty} (e^{-x} + 2 \cos 3x)$;

c) $\lim_{x \rightarrow \infty} [\ln(2+x) - \ln(1+x)]$;

d) $\lim_{x \rightarrow \infty} \arctan(e^x)$.

Problem 11. Find the horizontal and vertical asymptotes of the following curves.

a) $y = \frac{1+x^4}{x^2-x^4}$;

b) $y = \frac{x^3-x}{x^2-6x+5}$;

c) $y = \frac{2e^x}{e^x-5}$.

REFERENCES

- [1] J. Stewart: *Single Variable Calculus* 8th Edition, Cengage Learning, Boston 2015.