Intro to Generating Functions

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April 10, 2016

Outline

Ordinary Generating Functions (OGF)

Definition

The formal series

$$F(x) := \sum_{n \ge 0} f(n) x^n \in \mathbb{C}[[x]]$$

associated to the counting map $f : \mathbb{N}_0 \to \mathbb{C}$ is called *ordinary generating function*. We also use the notation $[x^n]F(x) := f(n)$.

Sum and Product of OGF

We recall that $\mathbb{C}[[x]]$ is an integral domain (actually a PID). We can perform sums and multiplications of OGF according to the way we sum and multiply elemnents in $\mathbb{C}[[x]]$.

Definition

Let $F(x) = \sum_{n \ge 0} f(n)x^n$ and $G(n) = \sum_{n \ge 0} g(n)x^n$ be two OGF. Then their *sum* is the OGF

$$F(x) + G(x) := \sum_{n \ge 0} (f(n) + g(n))x^n,$$

and their *product*, also called their *convolution* is the OGF

$$F(x)G(x) := \sum_{n \ge 0} \Big(\sum_{k=0}^{n} f(k)g(n-k) \Big) x^n.$$

Example: The OGF of the Fibonacci sequence is $F(x) = \frac{1}{1-x-x^2}$.

The Inverse of an OGF

Theorem

An OGF $F(x)$ is invertible (meaning that there exists an OGF $G(x)$ such that $F(x)G(x) = 1$) iff is $F(0) \neq 0$.

Proof: If $G(x)$ is the inverse of $F(x)$ then $F(0)G(0) = 1$ and so $F(0) \neq 0$. If $F(0) \neq 0$ then $G(0) = F(0)^{-1}$, and we can recurrently define the remaining coefficients of $G(0)$ by using the multiplication formula for OGF. □

Example: The OGF $F(x) = 1 - x$ satisfies that $F(0) = 1 \neq 0$. Therefore, it is invertible. The inverse of $F(x)$ is $G(x) = \sum_{n \geq 0} x^n$.

Convergence of OGF

Definition

The *degree* of an OGF $F(x) = \sum_{n \geq 0} f(n)x^n$, denoted by $\deg F(x)$ is the smallest n such that $f(n) \neq 0$. A sequence of OGF $\{F_i(x)\}_{i \in \mathbb{N}}$ *converges* to the OGF $F(x)$ if

$$\lim_{i \to \infty} \deg(F(x) - F_i(x)) = \infty.$$

We say that the infinite sum $\sum_{i \geq 0} F_i$ *coverges* to the OGF $F(x)$ if the sequence of partial sum converges to $F(x)$.

Theorem

The infinite series $\sum_{i \geq 0} F_i(x)$ converges iff $\lim_{i \to \infty} \deg F_i(x) = \infty$.

Proof: It follows directly from the definition of convergence. □

Composition of OGF

Definition

Let $F(x) = \sum_{n \ge 0} f(n)x^n$ and $G(n) = \sum_{n \ge 0} g(n)x^n$ be two OGF such that $G(0) = 0$. Then the composition of $F(x)$ and $G(x)$ is the OGF

$$F(G(x)) := \sum_{n \ge 0} f(n)G(x)^n.$$

Remarks:

- Notice that the condition $G(0) = 0$ guarantees that $\sum_{n \ge 0} f(n)G(x)^n$ converges. This is because $\deg G(x)^n \ge n \deg G(x)$, and so $\lim_{i \to \infty} \deg(f(i)G(x)^i) = \infty$.
- The expression $e^{1+x} = \sum_{n \ge 0}(x+1)^n/n!$ is not a valid OGF because the series does not converge in the sense we defined above.

A Few Popular OGFs

Theorem

The following are popular and useful OGFs:

1. $\sum_{n\geq 0} x^n = \frac{1}{1-x}$,
2. $\sum_{n\geq 0}(-1)^n x^n = \frac{1}{1+x}$,
3. $\sum_{n\geq 0} x^{2n} = \frac{1}{1-x^2}$,
4. $\sum_{n\geq 0} \binom{m}{n} x^n = (1+x)^m$,
5. $\sum_{n\geq 0} \binom{n+m}{n} x^n = \frac{1}{(1-x)^{m+1}}$,
6. $\sum_{n\geq 0} \binom{n}{m} x^n = \frac{x^m}{(1-x)^{m+1}}$.

Sketch of Proof: Pending...

Exponential Generating Functions

Definition

The formal series

$$F(x) := \sum_{n \geq 0} f(n) \frac{x^n}{n!}$$

associated to the counting map $f \colon \mathbb{N}_0 \to \mathbb{C}$ is called the *exponential generating function* (EGF) of f. We also use the notation $[x^n/n!]F(x) := f(n)$.

Sum, Product, and Composition of EGFs

Definition

Let $F(x) = \sum_{n \geq 0} f(n)x^n/n!$ and $G(n) = \sum_{n \geq 0} g(n)x^n/n!$ be two EGF. Then their *sum* is the EGF

$$F(x) + G(x) := \sum_{n \geq 0} (f(n) + g(n))x^n/n!,$$

and their *product*, also called their *convolution* is the OGF

$$F(x)G(x) := \sum_{n \geq 0} \Big(\sum_{k=0}^{n} \binom{n}{k} f(k)g(n-k) \Big) x^n/n!.$$

If, in addition, $G(0) = 0$, we composition of the EGFs F and G is given by

$$F(G(x)) := \sum_{n \geq 0} f(n)G(x)^n/n!.$$

The formal derivative of a GF

Definition

- The *derivative* of the OGF $F(x) = \sum_{n \geq 0} f(n)x^n$ is $F'(x) := \sum_{n \geq 0} nf(n)x^{n-1}$.
- The *derivative* of the EGF $F(x) = \sum_{n \geq 0} f(n)x^n/n!$ is $F'(x) := \sum_{n \geq 0} f(n)x^{n-1}$.

Theorem

Let $F(x)$ and $G(x)$ be two OGF (EGF) then the following hold:

- $(F(x) + G(x))' = F'(x) + G'(x)$,
- $F(x)G(x) = F'(x)G(x) + F(x)G'(x)$,
- $(F(G(x))' = F'(G(x))G'(x)$.

The formal derivative of a GF

Theorem

Let $F(x)$ and $G(x)$ be two OGFs such that $F(0) = 1$ and $G(0) = 0$. If $G'(x) = F'(x)/F(x)$ then $F(x) = \exp(G(x))$, where $\exp(G(x)) = \sum_{n \geq 0} \frac{G(x)^n}{n!}$.

Sketch of Proof: Pending...

Example 1: The EGF of the function $f : \mathbb{N} \to \mathbb{C}$ given by $f(0) = 1$ and $f(n+1) = f(n) + nf(n-1)$ if $n \geq 0$

General Newton Coefficients

Definition

- For $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}_0$, set $\binom{\lambda}{k} := \lambda(\lambda - 1) \dots (\lambda - k + 1)$.
- For an OGF $F(x)$ such that $F(0) = 0$, we define:

$$(1 + F(x))^{\lambda} := \sum_{n \geq 0} \binom{\lambda}{n} F(x)^n.$$

Example 2: We want to find all $f : \mathbb{N}_0 \to \mathbb{R}$ satisfying

$$\sum_{k=0}^{n} f(k) f(n-k) = 1.$$

A Practice Example

Example: Suppose that the function $f\colon \mathbb{N} \to \mathbb{C}$ has EGF $F(x) = e^{x+x^2/2}$.

- Find a recurrence formula for f.
- Find an explicit formula for f.

Rational Generating Functions

Theorem

Let $f : \mathbb{N}_0 \to \mathbb{C}$ *and* $Q(x) = 1 + c_1 x + \cdots + c_d x^d = \prod_{i=1}^{k} (1 - \alpha_i x)^{d_i}$, *where* $c_1, \dots, c_d \in \mathbb{C}$ *(*$c_d \neq 0$*). Then TFAE:*

1. $f(n+d) + c_1 f(n+d-1) + \cdots + c_d f(n) = 0$ *for every* $n \in \mathbb{N}_0$*;*
2. $F(x) = \sum_{n \ge 0} f(b) x^n = P(x)/Q(x)$, *where* $\deg P(x) < d$*;*
3. $F(x) = \sum_{n \ge 0} f(n) x^n = \sum_{i=1}^{k} g_i(x)/(1 - \alpha_i x)^{d_i}$*;*
4. $f(n) = \sum_{i=1}^{k} p_i(n) \alpha_i^n$, *where* $p_i(n)$ *is a polynomial in* n *such that* $\deg p_i < d_i$ *for each* $i \in \{1, \dots, k\}$*.*

Sketch of Proof: For each $i \in \{1, 2, 3, 4\}$, define the complex space

$$V_i := \{f : \mathbb{N}_0 \to \mathbb{C} \mid f \text{ satisfies } (i)\}.$$

Check that $\dim V_i = d$ for each i. Use this to check that $V_1 = V_2$ and $V_3 = V_4$. Finally, show that $V_3 \subseteq V_2$. $\square$

Rational Generating Functions (continuation)

Definition

A generating function $F(x) = \sum_{n \geq 0} f(n)x^n$ satisfying any of the four condition in the previous theorem is called a (proper) *rational* generating function.

Examples

Example 1: Let $f(n)$ be the number of paths with n non-intersecting steps starting from $(0,0)$ with directions east, north, or west.

1. Find the generating function of F of f.
2. Find a close formula for f.

Hint: Count the paths of length n ending in EE, WW, and NE.

Example 2: Write $(\sqrt{2}+\sqrt{3})^{1980}$ in decimal form. What is the last digit before and the first digit after the decimal point?
Hint: Compute the generating function of $(\sqrt{2}+\sqrt{3})^{2n}$.

Putting Structures on Finite Sets

Theorem

Let $f_1, \dots, f_n \colon \mathbb{N}_0 \to \mathbb{C}$, and denote by $E_{f_i}(x)$ the EGF of f_i. For every finite set S, let

$$h(|S|) = \sum_{(T_1,\dots,T_n)} f_1(|T_1|) \dots f_n(|T_n|),$$

where the sum runs over every ordered n-partition of S. Then the EGF $E_h(x)$ of h satisfies that $E_h(x) = E_{f_1}(x) \dots E_{f_n}(x)$.

Sketch of Proof: Suppose first that $n = 2$. If $|S| = s$, the fact that there are $\binom{s}{k}$ ordered partitions (T_1, T_2) such that $|T_1| = k$ of S implies that

$$h(s) = \sum_{k=0}^{s} \binom{s}{k} f_1(k) f_2(n-k).$$

Then $E_h(x) = E_{f_1}(x) E_{f_2}(x)$. Now extend to n by induction. $\square$.

The Compositional Formula

Theorem

Given $f\colon \mathbb{N} \to \mathbb{C}$ and $g\colon \mathbb{N}_0 \to \mathbb{C}$ with $g(0) = 1$, and for every finite set S let

$$h(0) = h([n]) = \sum_{\{T_1,\dots,T_k\}\in\pi([n])} f(|T_1|)\dots f(|T_k|)g(k)$$

if $|S| > 0$ and $h(0) = 1$. Then $E_h(x) = E_g(E_f(x))$.

Sketch of Proof: Defining, for every $k \in \{1, \dots, n\}$

$$h_k(n) = \frac{1}{k!}\sum_{(T_1,\dots,T_k)} f(|T_1|)\dots f(|T_k|)g(k),$$

we have $h(n) = \sum h_k(n)$. By the previous theorem, $E_{h_k}(x) = g(k)/k!E_f(x)^k$. Hence

$$E_h(x) = \sum_{k\geq 1} g(k)\frac{E_f(x)^k}{k!} = E_g(E_f(x)). \quad \square$$

The Compositional Formula: An Example

Example: In how many ways $h(n)$ we can form n people into nonempty lines, and then arrange these lines in a circular order?

Explanation: Let $f(n)$ and $g(n)$ the number of ways to form n people in a line and in a circle, respectively. Then $f(n) = n!$ and $g(n) = (n-1)!$. So the EGFs of f and g are

$$E_f(x) = \sum_{n \ge 1} x^n = \frac{x}{1-x} \quad \text{and} \quad E_g(x) = \sum_{n \ge 1} \frac{x^n}{n} = \ln(1-x)^{-1}.$$

Hence, using the previous theorem,

$$E_h(x) = E_g(E_f(x)) = \ln(\frac{1-x}{1-2x}) = \sum_{n \ge 1} (2^n - 1)(n-1)! \frac{x^n}{n!}.$$

Thus $h(n) = (2^n - 1)(n-1)!$. □

References

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