Lattices: An Introduction

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Outline

Joins and Meets

Definition (Joins and Meets)

Let P be a poset.

- If $S \subseteq P$ a *join* (or *supremum*) of S, denoted by
$$\bigvee_{s \in S} s,$$
is an element of $u \in P$ that is an upper bound of S satisfying that if u' is any other upper bound of S, then $u \leq u'$.
- The definition of a *meet* (or *infimum*) of $S \subseteq P$, denoted by
$$\bigwedge_{s \in S} s,$$
is dual to the definition of join.

Remark: Note that if a join (resp., meet) exists then it is unique.

Definition of Lattice

Definition

- A *join-semilattice* (resp., *meet-semilattice*) is a poset such that any pair of elements have a join (resp., meet).
- A *lattice* is a poset that is both a join-semilattice and a meet-semilattice.
- If L is a lattice and $S \subset L$ such that $r \vee s, r \wedge s \in S$ for all $r, s \in S$, we say that S is a *sublattice* of L.

Example of lattices:

- Every totally ordered set is a lattice.
- If L and M are lattices, so are $L^*, L \oplus M$, and $L \times M$. While $L + M$ is not a lattice, at least L or M is empty, $(L + M) \cup \{\hat{0}, \hat{1}\}$ is always a lattice.
- The lattices L and M are sublattices of $L \oplus M$ and $(L + M) \cup \{\hat{0}, \hat{1}\}$.

Complete Lattices

Theorem

If L is a lattice, the join (resp., the meet) of any finite subset of L exists.

Proof: It follows by induction. □

Remark:

- If L is a lattice and $S \subseteq L$ is an infinite subset, the join (resp., meet) of S might not exists. Consider an open interval of $\mathbb{R}$.
- A finite lattice always contains $\hat{0}$ and $\hat{1}$.

Definition

A lattice L is said to be *complete* if every subset of L has a join and a meet.

Lattices (continuation)

Theorem

A finite join-semilattice containing $\hat{0}$ is a lattice.

Sketch of proof: Let L be a join-semilattice with $\hat{0}$. For $u, v \in L$ consider the set $S = \{ s \in L \mid s \leq u \text{ and } s \leq v \}$. Take m to be the join of the finitely many elements of S. Check that $m = u \wedge v$.

□

Remark: The above theorem fails when the join-semilattice is not finite. Example?

Morphism of Lattices

Definition

Let $\varphi\colon L \to L'$ be map between lattices.

- φ is a *lattice homomorphism* if $\varphi(r \vee s) = \varphi(r) \vee \varphi(s)$ and $\varphi(r \wedge s) = \varphi(r) \wedge \varphi(s)$ for all $r, s \in L$.
- A lattice homomorphism $\varphi\colon L \to L'$ is said to be an *isomorphism* if it is a bijection. In this case, we say that the lattices L and L' are *isomorphic*.

Remarks:

- It is easy to check that a homomorphism of lattices is an order-preserving map.
- A homomorphism of lattices $\varphi\colon L \to L'$ is an isomorphism iff there exists a homomorphism of lattices $\psi\colon L' \to L$ such that $\varphi \circ \psi = \mathrm{Id}_{L'}$ and $\psi \circ \varphi = \mathrm{Id}_L$.

Finite Semimodular Lattices

Theorem

For a finite lattice L, the following conditions are equivalent:

1. *If $r, s \in L$ both cover $r \wedge s$ then $r \vee s$ covers both r and s.*
2. *L is graded, and the rank function ρ of L satisfies*

$$\rho(r) + \rho(s) \geq \rho(r \vee s) + \rho(r \wedge s) \quad \textit{for all } r, s \in L.$$

Sketch of proof: (2) $\implies$ (1): If r and s cover $r \wedge s$ then $\rho(r) = \rho(s) = \rho(r \wedge s) + 1$ and $\rho(r \vee s) > \rho(r) = \rho(s)$. Apply inequality (2).

(1) $\implies$ (2): If L is not graded there is a nongraded interval $[u, v]$ with minimal length. Take r_1, r_2 covering u such that $[r_1, v]$ and $[r_2, v]$ are both graded with different lengths. The saturated chains $r_i < r_1 \vee r_2 = t_1 < \cdots < t_n = v$ have lengths n. Contradiction.

If the inequality in (2) does not hold, take $r, s \in L$ with $(\ell(r \wedge s, r \vee s), \rho(r) + \rho(s))$ minimal (lexicographically) such that $\rho(r) + \rho(s) < \rho(r \wedge s) + \rho(r \vee s)$. If $s \wedge r < s' < s$, take the pair $(R, S) = (s' \vee r, s)$ to contradict the minimality of the pair (r, s).

□

Finite Semimodular Lattices

Definition (Modular Lattice)

- A lattice satisfying any of the above conditions is said to be *finite upper semimodular*. A finite lattice is called *lower semimodular* if its dual is upper semimodular.
- A lattice L is *modular* if it is upper and lower semimodular.

Example of modular lattices:

1. For every $n \in \mathbb{N}$, the lattice $[n]$ is modular.
2. If S is finite then $\mathcal{P}(S)$ is modular.
3. For which sets S is the lattice $\prod_S$ modular?
4. Give an example of lower semimodular lattice that is not upper semimodular.

A Characterization of Modular Lattices

Theorem (Characterization of Modular Lattices)

A lattice L is modular iff $r \vee (t \wedge s) = (r \vee t) \wedge s$ for all $r, s, t \in L$ such that $r \leq s$.

Sketch of proof: (Sufficiency) Since $r \leq s$ we have $r \vee (t \wedge s) \leq (r \vee t) \wedge s$. Using the modularity condition we can verify that $\rho(r \vee (t \wedge s)) = \rho((r \vee t) \wedge s)$. Hence $r \vee (t \wedge s) = (r \vee t) \wedge s$, as desired.

(Necessity) Take $r, s \in L$ both covering $r \wedge s$. Take $u \in L$ such that $r \leq u < r \vee s$, and let $v = u \wedge s$. Since $r \wedge s \leq v \leq s$ and $v \neq s$ (otherwise $r \vee s \leq u$), we have $v = r \wedge s$. Using $r \leq u$, we get

$$r = r \vee (r \wedge s) = r \vee (s \wedge u) = (r \vee s) \wedge u = u.$$

Hence L is upper semimodular. Dualizing we have $r \vee_* (t \wedge_* s) = (r \vee_* t) \wedge_* s$ for all $r, s, t \in L^*$ with $r \leq_* s$. Hence L is also lower semimodular, and so a modular lattice. □

Complemented and Atomic Lattices

Definition (Complemented Lattice)

A lattice L having $\hat{0}$ and $\hat{1}$ is said to be *complemented* if for all $r \in L$ there exists a *complement* $s \in L$, meaning $r \wedge s = \hat{0}$ and $r \vee s = \hat{1}$. If each $t \in L$ has a unique complement we say that L is *uniquely* complemented.

Example of complemented lattices: For any set S its power is (uniquely) complemented. A totally ordered set T is complemented iff $|T| \leq 2$. When is D_n complemented?

Definition

If L is a finite lattice with $\hat{0}$, an element $r \in L$ is an *atom* if it covers $\hat{0}$. If every element of L is the join of atoms, then L is said to be *atomic*. Dually, we can define *coatom* and *coatomic* lattice.

Example of atomic lattices: $\mathcal{P}(S)$ is atomic. $[n]$ is atomic iff $n \leq 2$.

Finite Geometric Lattice

Definition

If for all $r, s \in L$ such that $r \leq s$, the interval $[r, s]$ is itself complemented, we say that L is *relatively complemented*.

Theorem

For a finite upper semimodular lattice L the following conditions are equivalent.

1. *L is atomic.*
2. *L is relatively complemented.*

Proof: Omitted.

Definition (Finite Geometric Lattice)

A finite semimodular lattice satisfying the above conditions is called *finite geometric lattice*.

Distributive Lattices

Definition (Distributive Lattice)

A lattice L is said to be distributive if the following conditions hold:

- $r \wedge (s \vee t) = (r \wedge s) \vee (r \wedge t)$ for all $r, s, t \in L$;
- $r \vee (s \wedge t) = (r \vee s) \wedge (r \vee t)$ for all $r, s, t \in L$.

Remark: Both conditions in the above definition are equivalent.

Distributive Lattices (Examples)

Examples of distributive lattices:

1. Every totally ordered set is a distributive lattice.
2. For a set S its power set, $\mathcal{P}(S)$ is a distributive lattice.
3. For $n \in \mathbb{N}$, the lattice D_n is distributive.
4. If $n \geq 3$ the lattice $\prod_n$ is NOT distributive.

Distributive Lattices (continuation)

Theorem

For a distributive lattice L, the following conditions hold.

1. *L is modular.*
2. *The complement of each element, if exists, must be unique.*

Sketch of proof:

1. For $r, t, s \in L$ such that $r \leq s$ we have

$$r \vee (t \wedge s) = (r \vee t) \wedge (r \vee s) = (r \vee t) \wedge s.$$

   Therefore, by the characterization theorem of modular lattices, part (1) follows.

2. Let s such that $r, r' \in P$ are two complements of s. Then

$$r = r \vee ((s \wedge r) \vee (s \wedge r')) = r \vee (s \wedge (r \vee r')) = (r \vee s) \wedge (r \vee r') = r \vee r',$$

   and so $r' \leq r$. Similarly $r \leq r'$. Hence $r = r'$.

□

Antichains and Order Ideals

Definition

Let P be a poset.

1. A subset A of P is an *antichain* if any two distinct elements of A are incomparable.
2. A subset I of P is an *order ideal* if $s \in I$ and $r \leq s$ implies that $r \in I$.

Theorem

Let P be a finite poset. There is a bijection between the set of antichain and the set of order ideals of P.

Sketch of proof: Assign to the order ideal I the antichain A_I consisting of all maximal elements of I. Conversely, assign to the antichain A the order ideal $I_A := \{ s \in P \mid s \leq a \text{ for some } a \in A \}$.

□

The Lattice of Order Ideals

Definition

Let P be a poset.

1. An order ideal I is *generated* by the antichain A, if $I = I_A$.
2. If I is generated by $\{s\}$ it is called *principal* and denoted by Λ_s.
3. The set of all order ideals of P is denoted by $J(P)$.

Theorem

If P is a poset $J(P)$ is a distributive lattice.

Sketch of proof: $J(P)$ is a poset under inclusion. If I and J are order ideals of P then so are $I \cap J$ and $I \cup J$; therefore $J(P)$ is a lattice. Since intersection and union of sets distribute with each other, $J(P)$ is distributive. □

Join-irreducible Elements

Definition

Let L be a lattice and $s \in L$. We call s *join-irreducible* if $s \neq \hat{0}$ and s is not the join of two strictly smaller elements.

Theorem

Let P be a finite poset.

1. *An order ideal I of P is join-irreducible in $J(P)$ iff it is principal.*
2. *The set of join-irreducible of $J(P)$, considered as a subposet of $J(P)$, is isomorphic to P. Hence $J(P) \cong J(Q)$ iff $P \cong Q$.*

Proof: Straightforward. □

Fundamental Theorem of Finite Distributive Lattices

Theorem (Fundamental Theorem of FDL)

Let L be a finite distributive lattice (FDL). Then, up to isomorphism, there is a unique poset P such that $L \cong J(P)$.

Sketch of proof: Let P be the set of join-irreducibles of L. For $t \in L$ set $I_t = \{ s \in P \mid s \le t \}$. Define $\phi \colon L \to J(P)$ by $\phi(t) = I_t$. Since I_t is an order ideal for each $t \in L$, the map ϕ is well defined. The fact that $J(P)$ is a lattice implies that ϕ is injective. To show that ϕ is surjective, take $I \in J(P)$ and check that $\phi(t) = I$, where $t = \vee_{s \in I} s$. Check that $I = I_t$. The inclusion $I \subseteq I_t$ follows immediately. Conversely, take $u \in I_t$. Since $\vee_{s \in I} s = \vee_{s \in I_t} s = t$ we have

$$u = \vee_{s \in I_t} u \wedge s = \vee_{s \in I} u \wedge s.$$

Since u is join-irreducible $u \wedge s = u$ for some $s \in I$. Then $u \le s$, which means that $u \in I$. Therefore $I_t = I$ and so ϕ is onto. $\square$

FTFDL (consequences)

Theorem

If P is a poset of order n then $J(P)$ is graded of rank n. Furthermore, if $I \in J(P)$ then $\rho(I) = |I|$.

Proof: Exercise. □

Theorem

If L is a FDL, the following conditions are equivalent.

1. *L is complemented.*
2. *L is relatively complemented.*
3. *L is atomic.*
4. *$\hat{1}$ is a join of atoms.*

Proof: Exercise. □

References

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