# Introduction to Symmetric Functions

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# Homogeneous Symmetric Functions

## Definition

Let $n \in \mathbb{N}_0$ and $R$ be a commutative ring with identity. A *homogeneous symmetric function of degree* $n$ over $R$ is a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha$$

satisfying that

1. $\alpha$ runs over all weak composition $(\alpha_1, \alpha_2, \dots)$ of $n$;
2. $c_\alpha \in R$ and $x^\alpha$ stands for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \dots$;
3. $f(x_{w(1)}, x_{w(2)}, \dots) = f(x_1, x_2, \dots)$ for every bijection $w\colon \mathbb{N} \to \mathbb{N}$.

# The Algebra of Symmetric Functions

**Algebraic remarks:**

- The set of all homogeneous symmetric functions of degree $n$, which is dented by $\Lambda_R^n$, is an $R$-module.
- Since $f \in \Lambda_R^n$ and $g \in \Lambda_R^m$ implies $fg \in \Lambda_R^{n+m}$, the vector space

$$\Lambda_R = \Lambda_R^0 \oplus \Lambda_R^1 \oplus \dots,$$

is, in fact, an $R$-algebra.

## Definition

Let $R$ be a commutative ring with 1. The algebra $\Lambda_R$ described before is called the *algebra of symmetric functions* over $R$.

# Notations Related to Partitions

## Definition

A *partition* $\lambda$ of $n \in \mathbb{N}_0$ is a decreasing sequence $(\lambda_1, \lambda_2, \dots)$ such that $\lambda_i \in \mathbb{N}_0$ and $\lambda_1 + \lambda_2 + \cdots = n$; we write $\lambda \vdash n$. The nonzero $\lambda_i$'s are called *parts* of $\lambda$. Note that there exists $N \in \mathbb{N}$ such that $\lambda_i = 0$ for all $i > N$. The following notations will be used:

- $\mathrm{Par}(n)$ is the set of partitions of $n$, and $p(n) := |\mathrm{Par}(n)|$;
- $\mathrm{Par} := \cup_{n \geq 0} \mathrm{Par}(n)$;
- if $\lambda \in \mathrm{Par}(n)$, we write $|\lambda| = n$;
- the length $\ell(\lambda)$ of $\lambda$ is the largest $k$ such that $\lambda_k \neq 0$;
- $m_i = m_i(\lambda)$ is the number of parts of $\lambda$ equaling $i$;
- $\lambda' \in \mathrm{Par}(n)$ given by $m_i(\lambda') = \lambda_i - \lambda_{i+1}$ is called the *conjugate* partition of $\lambda$.

# Monomial Symmetric Functions

## Definition

For $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$, set $m_\lambda = \sum_\alpha x^\alpha$, where $\alpha$ runs over all distinct permutations of the vector $(\lambda_1, \lambda_2, \dots)$. We call $m_\alpha$ *monomial symmetric function*.

## Theorem

*The set $\{m_\lambda \colon \lambda \vdash n\}$ is a basis for $\Lambda^n$.*

**Sketch of Proof:** Let $B = \{m_\lambda \colon \lambda \vdash n\}$. If $f = \sum_\alpha c_\alpha x^\alpha$, then

$$f = \sum_\alpha c_\alpha x^\alpha = \sum_{\lambda \vdash n} c_\lambda m_\lambda.$$

Therefore $B$ generates $\Lambda^n$. The linear independence is obvious. Hence $B$ is a basis for $\Lambda^n$ and, in particular, $\dim \Lambda^n = p(n)$. □

# Elementary Symmetric Functions

## Definition

For $\lambda = (\lambda_1, \lambda_2, \dots)$, if Par is not the empty partition, define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots, \text{ where } e_n = m_{1^n} = \sum_{i_1 < \dots < i_n} x_{i_1} \dots x_{i_n};$$

define $e_0 = m_\emptyset = 1$. We call $e_\lambda$ an *elementary symmetric function*.

For an infinite matrix $A = (a_{ij})_{i,j \ge 1}$ with only finitely many nonzero entries, set $r_i = \sum_j a_{ij}$ and $c_j = \sum_i a_{ij}$ and define

$$\text{row}(A) = (r_1, r_2, \dots),$$
$$\text{col}(A) = (c_1, c_2, \dots).$$

# Elementary Symmetric Functions

## Theorem

*Let $\lambda \vdash n$ and $\alpha = (\alpha_1, \alpha_2, \dots)$ be a weak composition of n. Then the coefficient $M_{\lambda\alpha}$ of $x^\alpha$ in $e_\lambda$ equals the number of binary matrices B such that row$(B) = \lambda$ and col$(B) = \alpha$.*

**Sketch of Proof:** Consider the matrix of formal variables $X = (x_{ij})_{i,j \geq 1}$, where $x_{ij} = x_j$ for every $i$. Let $e_\lambda = \sum_\alpha M_{\lambda\alpha} x^\alpha$. A monomial $x^\alpha$ of $e_\lambda$ is obtained by choosing, for each $i$, $\lambda_i$ entries of the $i$th row of $X$ and, then, multiplying all the chosen entries. Thus, the coefficient $M_{\lambda\alpha}$ of $x^\alpha$ equals the number of binary matrices $B$ such that row$(B) = \lambda$ and col$(B) = \alpha$. Conversely, each of such binary matrices determines a monomial $x^\alpha$ of $e_\lambda$. □

## Corollary

*If $\lambda, \mu \vdash n$, then $M_{\lambda\mu} = M_{\mu\lambda}$.*

**Sketch of Proof:** Just note $e_\lambda = \sum_\alpha M_{\lambda\alpha} x^\alpha = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu$.

□

# Elementary Symmetric Functions (continuation)

**Proposition**

$\prod_{i,j}(1 + x_i y_j) = \sum_{\lambda,\mu} M_{\lambda\mu} m_\lambda(x) m_\mu(y)$.

**Sketch of Proof:** Pending... □

**Definition**

Let $n \in \mathbb{N}$. For $\mu, \lambda \in \mathrm{Par}(n)$, we write $\mu \leq \lambda$ if

$$\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \text{ for every } i \geq 1.$$

The relation $\leq$ is a partial order in $\mathrm{Par}(n)$ that is called *dominance order*.

# Fundamental Theorem of Symmetric Functions

**Theorem (Fundamental Theorem of Symmetric Functions)**

*The set* $\{e_\lambda \mid \lambda \vdash n\}$ *is a basis for* $\Lambda^n$. *Equivalently,* $e_1, e_2, \ldots$ *are algebraically independent and generated* $\Lambda$ *as a* $\mathbb{Q}$*-algebra.*

**Sketch of Proof:** Pending... □

# Complete Homogeneous Symmetric Functions

**Definition**

Pending...

# Power Sum Symmetric Functions

**Definition**

Pending...

# Specializations

**Definition**

Pending...

# Scalar Product

## Definition

Pending...

# Schur Functions

## Definition

Pending...

# References

M. Bona. *A Walk Through Combinatorics*. World Scientific, New Jersey, 2012.

G. Grätzer. *Lattice Theory*. Dover Publication, Inc., New Jersey, 2009.

R. Stanley. *Enumerative Combinatorics*. Cambridge University Press, New York, 2012.

J. H. van Lint and R. M. Wilson. *A Course in Combinatorics*. Cambridge University Press, Cambridge, 1994.